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UNIVERSITY OF ALBERTA

NONOSCILLATION OF LINEAR DIFFERENCE SYSTEMS

by

RENÉ NESTOR LAFAUT

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
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IN
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DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

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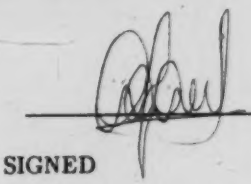

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And though I have the gift of prophecy,
and understand all mysteries, and all knowledge
and have all faith, so as to remove mountains,
and have not love, I am nothing. 1Cor13:2

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled NONOSCILLATION OF LINEAR DIFFERENCE SYSTEMS submitted by RENÉ NESTOR LAFAUT in partial fulfillment of the degree of Master of Science.

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ABSTRACT

In this thesis we develop a theory of nonoscillation for linear difference systems which is analogous to that for linear differential systems. A geometric approach is presented for determining best possible bounds on the operators involved which guarantee the nonoscillation of the system. We show that many of the results that hold for differential systems also hold for difference systems.

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CHAPTER I

INTRODUCTION

Let A and f be $n \times n$ and $n \times 1$ matrix functions respectively that are piecewise continuous on a real interval I. Let x be a $n \times 1$ matrix function. If $f \neq 0$ the equation

$$x' = Ax + f \tag{1.1}$$

is called a linear nonhomogeneous differential system. If $f = 0$ then (1.1) becomes

$$x' = Ax \tag{1.2}$$

and is called a linear homogeneous differential system. A function x is called a solution to (1.1) if it is piecewise smooth and satisfies (1.1) on I except at the points where A and f are discontinuous. If $u \in \mathbb{R}^n$ and $t_0 \in I$, the problem of finding a solution to (1.1) with the condition $x(t_0) = u$ is called an initial value problem (IVP). If $t_1, \ldots, t_n \in I$, the problem (1.1) with the condition $x_j(t_j) = u_j$ is called a boundary value problem (BVP).

It is well known that the only solution of (1.2) satisfying $x(t_0) = 0$ is the zero solution, $x = 0$. From the linearity of the situation, this implies that the IVP (1.1), $x(t_0) = u$, has a unique solution. Similarly, if $x = 0$ is the only solution of the homogeneous BVP (1.2), $x_j(t_j) = 0$, then the BVP (1.1), $x_j(t_j) = u_j$, has a unique solution. This is certainly the case if the points $t_1, \ldots, t_n$ are sufficiently close together, but it is not necessarily so otherwise. It

is therefore of interest to obtain conditions which guarantee that $x = 0$ is the only solution of the homogeneous BVP for all choices of $t_1, \ldots, t_n \in I$.

A nonzero piecewise smooth vector function is called oscillatory on I if each of its components has a zero on I. Equation (1.2) is said to be oscillatory on I if there exists an oscillatory solution to (1.2) on I. Suppose that the BVP (1.1) describes a physical system where $x_j(t_j) = u_j$ are measured quantities at time $t_j \in I$. Then the associated system (1.2) is nonoscillatory on I if and only if for every vector function f and boundary conditions $x_j(t_j) = u_j$, there exists a unique solution x to the BVP (1.1).

The main studies on nonoscillation of differential systems are due to Nehari [2],Schwarz[3], and Friedland [1]. A comprehensive bibliography may be found in the monograph [1] of Friedland. In particular, Nehari and Friedland investigate best possible (or largest) constants $c_{1,2}$ such that

$$\int_I |A|_{1,2} < c_{1,2}$$

implies (1.2) is nonoscillatory on I. Here $|\cdot|_{1,2}$ denotes the matrix norm associated with the vector norms $|\cdot|_1, |\cdot|_2$ defined by

$$|A|_{1,2} = \sup\{\frac{|Ay|_2}{|y|_1},\ y \neq 0\}.$$

In part, Nehari shows that if A is continuous on $I = [a, b]$ and $|\cdot|_1 = |\cdot|_2 = \|\cdot\|$ is the Euclidean norm, then $c_{1,2} = \frac{\pi}{2}$. To show this, he takes two

nonzero arbitrary solutions y and w of the systems

$$y' = Ay \tag{1.3}$$

$$w' = Bw \tag{1.4}$$

and projects them onto the unit ball

$$u = \frac{y}{\|y\|}$$
$$v = \frac{w}{\|w\|}$$

to show that, if C is an arbitrary orthogonal $n \times n$ matrix, then

$$(vCu)' \leq (\|A\| + \|B\|)(1 - (vCu)^2)^{\frac{1}{2}} \tag{1.5}$$

which implies

$$|\sin^{-1}[v(a)Cu(a)] - \sin^{-1}[v(b)Cu(b)]| \leq \int_I (\|A\| + \|B\|) . \tag{1.6}$$

Suppose that u is oscillatory on I. Then the system (1.3) is oscillatory. Moreover, without loss of generality, let t_j be a zero of u_j, for $j = 1, \ldots, n$ and $t_j \leq t_{j+1}$. Let $t_0 = a$, $t_{n+1} = b$, and

$$c_{i,k}(t) = \begin{cases} c_{i,k} & \text{if } t < t_k,\ 1 \leq k \leq n \\ -c_{i,k} & \text{if } t \geq t_k,\ 1 \leq k \leq n. \end{cases}$$

Observe that $vC(t)u$ is continuous on I, that $C(t)$ is constant on any interval (t_i, t_{i+1}), and that $C(a) = -C(b)$. Because $C(t)$ is constant on

any interval (t_i, t_{i+1}), (1.5) can be integrated on each interval (t_i, t_{i+1}) with $C(t)$ replacing C. Adding all the results up implies

$$|\sin^{-1}[v(b)Cu(b)] + \sin^{-1}[v(a)Cu(a)]| \leq \int_I (\|A\| + \|B\|).$$

This, together with (1.6), implies

$$|\sin^{-1}[v(b)Cu(b)]| + |\sin^{-1}[v(a)Cu(a)]| \leq \int_I (\|A\| + \|B\|).$$

Letting $B = 0$ in (1.4) and $v(a) = u(a)C^{-1}$ implies

$$\frac{\pi}{2} + |\sin^{-1}[v(a)Cu(b)]| \leq \int_I |A|$$

which proves the result. The constant $\pi/2$ is shown to be best possible by consideration of examples, which demonstrate that $\pi/2$ cannot be replaced by a larger constant.

Friedland obtains this result as a specific case of a unified approach to the subject. The approach leads to the study of related infimum problems. Let S denote the set of oscillatory vector functions that are piecewise smooth on I and let

$$c_{1,2} = \inf\{\int_I \frac{|x'|_2}{|x|_1} : x \in S\}.$$

Then the condition

$$\int_I |A|_{1,2} < c_{1,2}$$

implies that the system (1.2) is nonoscillatory on I. This is the case since, if x is any oscillatory solution to (1.2), then

$$c_{1,2} \leq \int_I \frac{|x'|_2}{|x|_1} \leq \int_I \frac{|Ax|_2}{|x|_1} \leq \int_I |A|_{1,2}.$$

This constant is shown to be best possible, and the constant is evaluated for many norms in $\mathbb{R}^2$. In many cases, he also shows that the constant is the same for higher dimensions.

In this thesis a parallel theory is developed for linear difference equations of the form

$$\Delta x = Ax \tag{1.7}$$

where A, and x are $n \times n$ and $1 \times n$ matrix functions respectively defined on $\mathcal{D}_N = \{0, 1, \ldots, N\}$, and Δ is the first forward difference operator defined by $\Delta x(t) = x(t+1) - x(t)$. A vector function x is called *oscillatory* on $\mathcal{D}_N$ if it is nonzero and each component either changes sign or has a zero on $\mathcal{D}_N$. The system (1.7) is called *oscillatory* on $\mathcal{D}_N$ if it has an oscillatory solution.

When $|A|_{1,2} \leq \delta$, a nonoscillation criterion for difference systems of the type (1.7) is established in the form

$$\sum_{0}^{N-1} |A|_{1,2} < k_{1,2} \tag{1.8}$$

where $k_{1,2} = k_{1,2}^{(n)}(N, \delta)$ is the best possible constant, and $\sum_0^{N-1} |A|_{1,2}$ denotes the sum $\sum_{t=0}^{t=N-1} |A(t)|_{1,2}$.

The condition $|A(t)| \leq \delta$ for $t \in \mathcal{D}_{N-1}$ is imposed to ensure a nontrivial theory. If $|A(t)|$ is not bounded, then there exist difference systems of the type (1.7) that are oscillatory on $\mathcal{D}_1$. For given $\delta > 0$, we require that N be sufficiently large to ensure that system (1.7) is oscillatory for some A with $|A(t)| \leq \delta$. In particular, if $\delta \to 0^+$, then $N \to \infty$. This issue is dealt with more fully in Theorem 2.1 of Chapter 2.

The outline of this thesis is as follows. In Chapter 2, the nonoscillation criterion just described is established and then strengthened for particular systems. In addition, the constant $k_{1,2}$ is also compared with $c_{1,2}$, the corresponding nonoscillation constant for differential systems.

In Chapter 3, the problem of determining $k_{1,2}$, where $|\cdot|_1$ and $|\cdot|_2$ are absolute symmetric norms, is considered. The approach developed is a geometrical one, and allows us to show, in part, that $k_{1,2}^{(2)} \to c_{1,2}$ if $\delta \to 0^+$. In Chapter 4 the problem of determining $k_{1,2}^{(n)}$ for $n \geq 3$ is considered, and under certain conditions we show that $k_{1,2}^{(n)} = k_{1,2}^{(2)}$.

CHAPTER II

GENERAL RESULTS FOR DIFFERENCE SYSTEMS

§2.1 Nonoscillation Criteria

Consider the difference equation

$$\Delta x = Ax \tag{2.1}$$

where A and x are respectively $n \times n$ and $n \times 1$ matrix functions with domain $\mathcal{D}_N = \{0, 1, \ldots, N\}$ and $\Delta x(t) = x(t+1) - x(t)$. A nonzero vector function x is called *oscillatory* on $\mathcal{D}_N$ if every component of x either changes sign or has a zero on $\mathcal{D}_N$. The system (2.1) is *oscillatory* on $\mathcal{D}_N$ if there exists an oscillatory solution to (2.1).

Let $|\cdot|_1$, $|\cdot|_2$ be a pair of norms, and let x be a nonzero solution to (2.1). Then (2.1) implies

$$\frac{|\Delta x|_2}{|x|_1} = \frac{|Ax|_2}{|x|_1} \leq \sup\{\frac{|Ay|_2}{|y|_1} : y \neq 0\} = |A|_{1,2} \tag{2.2}$$

so that

$$\sum_0^{N-1} \frac{|\Delta x|_2}{|x|_1} \leq \sum_0^{N-1} |A|_{1,2}. \tag{2.3}$$

Let $V_{1,2}^{(n)}(N, \delta)$, where $0 < \delta < 1$, denote the set of oscillatory vector functions $x : \mathcal{D}_N \to \mathbb{R}^n$ such that $|\Delta x(t)|_2 \leq \delta |x(t)|_1$ for each $t \in \mathcal{D}_{N-1}$. It follows from (2.2) that if

$$k_{1,2}^{(n)}(N, \delta) = \inf\{\sum_0^{N-1} \frac{|\Delta x|_2}{|x|_1} : x \in V_{1,2}^{(n)}(N, \delta)\},$$

then the condition

$$\sum_{0}^{N-1} |A|_{1,2} < k_{1,2}^{(n)}(N,\delta) \tag{2.4}$$

implies system (2.1) is nonoscillatory on $\mathcal{D}_N$.

We will usually denote $V_{1,2}^{(n)}(N,\delta)$ and $k_{1,2}^{(n)}(N,\delta)$ by $V_{1,2}$ and $k_{1,2}$ respectively except when it is necessary to emphasize the dimension of the underlying space $\mathbb{R}^n$, the domain $\mathcal{D}_N$, or the parameter δ.

The set $V_{1,2}^{(n)}(N,\delta)$ is nonempty, if $N \geq 2[\mu_2/(\mu_1\delta)+1]$ where

$$\mu_1 = \min\{|(\epsilon\delta_{j,1},\ldots,\epsilon\delta_{j,n})|_1 : j = 1,\ldots,n; \epsilon = \pm 1\},$$

$$\mu_2 = \max\{|(\epsilon\delta_{j,1},\ldots,\epsilon\delta_{j,n})|_2 : j = 1,\ldots,n; \epsilon = \pm 1\},$$

and $\delta_{j,k}$ is the Kronecker delta function defined by $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$. To see this, it suffices to consider the case where n=2 and

$$|x|_1 = \mu_1 \max\{|x_1|,|x_2|\}\ ,\ |x|_2 = \mu_2(|x_1|+|x_2|)$$

because

$$\mu_1 \max\{|x_1|,|x_2|\} \leq |x|_1\ ,\ |x|_2 \leq \mu_2(|x_1|+|x_2|)$$

imply

$$\frac{|\Delta x(t)|_2}{|x(t)|_1} \leq (\frac{\mu_2}{\mu_1})\frac{(|\Delta x_1(t)|+|\Delta x_2(t)|)}{\max\{|x_1(t)|,|x_2(t)|\}}\ .$$

Let x be defined by

$$x(t) = \begin{cases} (t(\mu_1\delta/\mu_2),1) & \text{if } t \leq n \\ (1,1) & \text{if } t = n+1 \\ (n(\mu_1\delta/\mu_2), n(N-t)(\mu_1\delta/\mu_2)^2) & \text{otherwise.} \end{cases}$$

where $n\delta < \mu_2/\mu_1 \le (n+1)\delta$, and $N = 2n+3$. Then clearly $x \in V_{1,2}^{(2)}(N,\delta)$ and $N \le 2[\mu_2/(\mu_1\delta)+1]$.

The constant in the nonoscillation criterion (2.4) cannot be replaced by a larger constant. To show this one reasons analogously to the proof of Theorem 3.1 from Friedland [1,p.14]. Assume that the norm $|\cdot|_1$ is uniformly Fréchet-differentiable (UF), that is $|x|_1$ is continuously differentiable on the Euclidean unit sphere. If $x \in V_{1,2}$, we construct a matrix function A such that both $\Delta x = Ax$, and $|A|_{1,2} = |x|_2/|x|_1$ hold. Let $\epsilon(t) = (\epsilon_1(t),\dots,\epsilon_n(t))$ where $\epsilon_i(t) = \frac{\partial}{\partial x_i}|x(t)|_1$. Then, since $|\cdot|_1$ is UF, we find

$$|\epsilon(t)|_1^* := \sup\{\frac{|\sum_j \epsilon_j y_j|}{|y|_1} : y \neq 0\} = \sum_{j=1}^{n} \frac{\epsilon_j(t)x_j(t)}{|x(t)|_1} = 1, \tag{2.5}$$

where $|\cdot|_1^*$ is called the conjugate norm of $|\cdot|_1$. Furthermore, let $a_{i,j} = [\Delta x_i(t)]\epsilon_j(t)/|x(t)|_1$. Then by (2.5)

$$\Delta x_i(t) = \Delta x_i(t)\{\sum_{j=1}^{n} \frac{\epsilon_j(t)x_j(t)}{|x(t)|_1}\} = \sum_{j=1}^{n} a_{i,j}(t)x_j(t)$$

so that x satisfies $\Delta x = Ax$. Moreover

$$|A(t)|_{1,2} = \sup\{\frac{|Ay|_2}{|y|_1} : y \neq 0\} = \frac{|\Delta x(t)|_2}{|x(t)|_1} \sup\{\frac{|\sum \epsilon_j y_j|}{|y|_1} : y \neq 0\} = \frac{|\Delta x(t)|_2}{|x(t)|_1}.$$

Since $x \in V_{1,2}$ is arbitrary and may be chosen so that $\sum_0^{N-1} |\Delta x|_2/|x|_1$ is as close as we please to $k_{1,2}$, it follows that $k_{1,2}$ is best possible when $|\cdot|_1$ is UF. If the norm $|\cdot|_1$ is not UF, then from Friedland [1,p.6], given any $\epsilon > 0$, there exists a UF norm $|\cdot|_3$ such that for each $x \in \mathbb{R}^n$

$$(1-\epsilon)|x|_3 \le |x|_1 \le (1+\epsilon)|x|_3. \tag{2.6}$$

Let $x \in V_{1,2}$. Then there exists an $n \times n$ matrix function A such that $\Delta x = Ax$ and

$$|A|_{3,2} = \frac{|\Delta x|_2}{|x|_3}. \tag{2.7}$$

Now by (2.6) we deduce that $|A|_{1,2} \le (1-\epsilon)^{-1}|A|_{3,2}$, so that (2.7) implies

$$|A|_{1,2} \le (1-\epsilon)^{-1}|A|_{3,2} = (1-\epsilon)^{-1}\frac{|\Delta x|_2}{|x|_3} \le \frac{(1+\epsilon)}{(1-\epsilon)}\frac{|\Delta x|_2}{|x|_1}.$$

This together with (2.2) implies there always exists a matrix function A which is as close to $|\Delta x|_2/|x|_1$ in norm as required satisfying $\Delta x = Ax$. This proves $k_{1,2}$ is best possible.

We summarize the preceding discussion in the following theorems.

THEOREM 2.1. *Let $\delta_{j,k}$ be the Kronecker delta function. Then*

(i) $\mu_1 = \min\{|(\epsilon\delta_{j,1},\dots,\epsilon\delta_{j,n})|_1 : j = 1,\dots,n : \epsilon = \pm 1\}$,

(ii) $\mu_2 = \max\{|(\epsilon\delta_{j,1},\dots,\epsilon\delta_{j,n})|_2 : j = 1,\dots,n : \epsilon = \pm 1\}$,

(iii) $N \ge 2[\mu_2/(\mu_1\delta) + 1]$.

imply $V_{1,2}^{(n)}(N,\delta)$ *is nonempty.*

THEOREM 2.2. *Suppose that*

(i) $|A(t)|_{1,2} \le \delta$ *for each* $t \in \mathcal{D}_N$,

(ii) $\sum_0^{N-1}|A|_{1,2} < k_{1,2}$,

where $k_{1,2} = \inf\{\sum_0^{N-1}|\Delta x|_2/|x|_1 : x \in V_{1,2}\}$.

Then the difference system (2.1) is nonoscillatory on $\mathcal{D}_N$. *The nonoscillation constant* $k_{1,2}$ *is best possible.*

Because we are only interested in $k_{1,2}$ when $V_{1,2}$ is nonempty, we let N satisfy condition (iii) in Theorem 2.1 for this thesis. With this restriction, it is also of interest to compare $k_{1,2}$ as $\delta \to 0^+$ with the corresponding nonoscillation constant $c_{1,2}$ for differential systems. To do this, we will consider the constant

$$\hat{c}_{1,2} = \lim_{\delta \to 0^+} k_{1,2}(N(\delta), \delta) \tag{2.8}$$

where $N(\delta) = 2[\mu_2/(\mu_1\delta) + 1]$, $[\,\cdot\,]$ is the greatest integer function, and μ_1 and μ_2 are defined in Theorem 2.1. Note that the reason for taking the $N = N(\delta)$ in (2.8), is that $k_{1,2}(N, \delta)$ as a function of N is nonincreasing.

Let P be a diagonal $n \times n$ matrix function whose entries are strictly positive on $\mathcal{D}_N$. Consider the change of variables $\boldsymbol{w} = P\boldsymbol{y}$ where $\boldsymbol{y}$ is any solution of (2.1). Then $\boldsymbol{w}$ satisfies the system

$$\Delta \boldsymbol{w}(t) = [P(t+1)A(t)P^{-1}(t) + P^{-1}(t)\Delta P(t)]\boldsymbol{w}(t). \tag{2.9}$$

A solution $\boldsymbol{w}$ of (2.9) is oscillatory if and only if $\boldsymbol{y} = P^{-1}\boldsymbol{w}$ is an oscillatory solution of (2.1). So, if the condition

$$\sum_{0}^{N-1} |P(t+1)A(t)P^{-1}(t) + P^{-1}(t)\Delta P(t)|_{1,2} < k_{1,2} \tag{2.10}$$

holds, then the difference system (2.1) is nonoscillatory. This criterion is analogous to a result for differential systems by Nehari [2]. By choosing P appropriately for particular $A(t)$, the left hand side of the inequality in (2.10) may be reduced to strengthen the nonoscillation criterion. For example, let $P(t) = (1-c)^t I$

where I is the identity matrix and $0 \le c \le 1$. Then the condition in (2.10) becomes

$$\sum_{0}^{N-1} |(1-c)A(t) - cI|_{1,2} < k_{1,2},$$

which essentially says that if $A(t)$ is close enough to cI, then system (2.1) is nonoscillatory.

This discussion is summarized by the following corollaries.

COROLLARY 2.3. *Suppose that*

(i) $|A(t)|_{1,2} \le \delta$, *for each* $t \in \mathcal{D}_N$,

(ii) P *is a diagonal matrix function whose entries are strictly positive on* $\mathcal{D}_N$,

(iii) $\sum_0^{N-1} |P(t+1)A(t)P^{-1}(t) + P^{-1}(t)\Delta P(t)|_{1,2} < k_{1,2}$.

Then the difference system (2.1) is nonoscillatory on $\mathcal{D}_N$. *The constant* $k_{1,2}$ *is best possible.*

COROLLARY 2.4. *Suppose that*

(i) $|A(t)|_{1,2} \le \delta$, *for each* $t \in \mathcal{D}_N$,

(ii) $\sum_0^{N-1} |(1-c)A(t) - cI|_{1,2} < k_{1,2}$, *where* $0 < c < 1$.

Then the difference system (2.1) is nonoscillatory on $\mathcal{D}_N$. *The constant* $k_{1,2}$ *is best possible.*

§2.2 Comparison of the constants $k_{1,2}$ and $c_{1,2}$

One might expect that $c_{1,2}$, the corresponding nonoscillation constant for differential systems of the type

$$x' = Ax \tag{2.11}$$

is equal to $\hat{c}_{1,2}$ defined in (2.8). To see this, observe that systems of type (2.1) defined on $\mathcal{D}_{N-1}$ satisfying $|A|_{1,2} < \delta$ on $\mathcal{D}_{N-1}$ may be written in the form

$$D_h \boldsymbol{x} = A\boldsymbol{x} \tag{2.12}$$

with domain $I_h = \{a, a+h, \ldots, b\}$ satisfying $|A(t)|_{1,2}h \leq \delta$ for $t \in I_h$ where D_h is the forward difference operator defined by $D_h\boldsymbol{x}(t) = \frac{1}{h}(\boldsymbol{x}(t+h) - \boldsymbol{x}(t))$, and $b - a = Nh$. From §2.1, the nonoscillation criteria for systems of the type (2.12) satisfying $|A(t)|_{1,2}h \leq \delta$ are

$$\sum_a^{b-h} |A(t)|_{1,2}h < k_{1,2},$$

$$\sum_a^{b-h} |P(t+h)A(t)P^{-1}(t) + P^{-1}(t)D_hP(t)|_{1,2}h < k_{1,2}, \tag{2.13}$$

$$\sum_a^{b-h} |(1-ch)A(t) - cI|_{1,2}h < k_{1,2} \quad \text{where} \quad 0 \leq ch < 1,$$

for Theorem 2.2, and Corollaries 2.3 and 2.4 respectively. If we let $\delta \to 0^+$, and $A(t)$ be such that $|A|_{1,2} \leq M$ for some $M \in \mathbb{R}$, then $h \to 0^+$ and the system in (2.12) becomes the one in (2.11). In this case, the nonoscillation criteria in (2.13) would suggest the following nonoscillation criteria

$$\int_a^b |A|_{1,2} < \hat{c}_{1,2},$$

$$\int_a^b |PAP^{-1} + P^{-1}P'|_{1,2} < \hat{c}_{1,2},$$

$$\int_a^b |A - cI|_{1,2} < \hat{c}_{1,2},$$

for the systems of type (2.11) where $\hat{c}_{1,2}$ is defined in (2.8). These criteria are confirmed by Nehari [2], and Friedland [1]; while we show that in fact $\hat{c}_{1,2} = c_{1,2}$ for some standard norms.

CHAPTER III

THE PROBLEM IN $\mathbb{R}^2$

§3.1 Geometric Considerations

For a polygon x in $\mathbb{R}^2$ with $N+1$ nonzero vertices $x(0), x(1), \ldots, x(N)$ we consider the sum

$$S_{1,2}x = \sum_{0}^{N-1} \frac{|\Delta x|_2}{|x|_1}. \tag{3.1}$$

We refer to the ray through O and $x(t)$ as the ray $Ox(t)$, and shall consider the problem of minimizing $S_{1,2}x$ over all $N-$gons with successive nonzero vertices on a given ordered set of rays.

First, observe that $S_{1,2}x$ is invariant with respect to replacing x by kx where k is a nonzero scalar. Thus there is no loss of generality in assuming $|x(t)|_1 = 1$ for particular $t \in \mathcal{D}_N$.

Consider the polygon $\hat{x}$ with the vertices $\hat{x}(0), \ldots, \hat{x}(N)$ where $\hat{x}(s) = x(s)$, for $s \in \mathcal{D}_t$, and $\hat{x}(s) = kx(s)$, for $s \in \mathcal{D}_N \backslash \mathcal{D}_t$. Then the sum

$$S_{1,2}\hat{x} = \sum_{0}^{N-1} \frac{|\Delta \hat{x}|_2}{|\hat{x}|_1} \tag{3.2}$$

differs from the sum $S_{1,2}x$ in (3.1) only in the terms corresponding to the edges $[\hat{x}(t), \hat{x}(t+1)] = [x(t), kx(t+1)]$ in (3.2), and $[x(t), x(t+1)]$ in (3.1) which are

$$|kx(t+1) - x(t)|_2 \quad , \quad |x(t+1) - x(t)|_2$$

respectively, if we assume $|x(t)|_1 = 1$. To minimize the contribution to the sum $S_{1,2}\hat{x}$ from this term, we should choose k so that the first expression is minimized. Effectively this means choosing $\hat{x}(t+1) = kx(t+1)$ so that the ray

$O\hat{x}(t+1)$ supports at $\hat{x}(t+1)$ a $|\cdot|_2$ – sphere centered on $\hat{x}(t) = x(t)$ as illustrated in Figure 3.3.

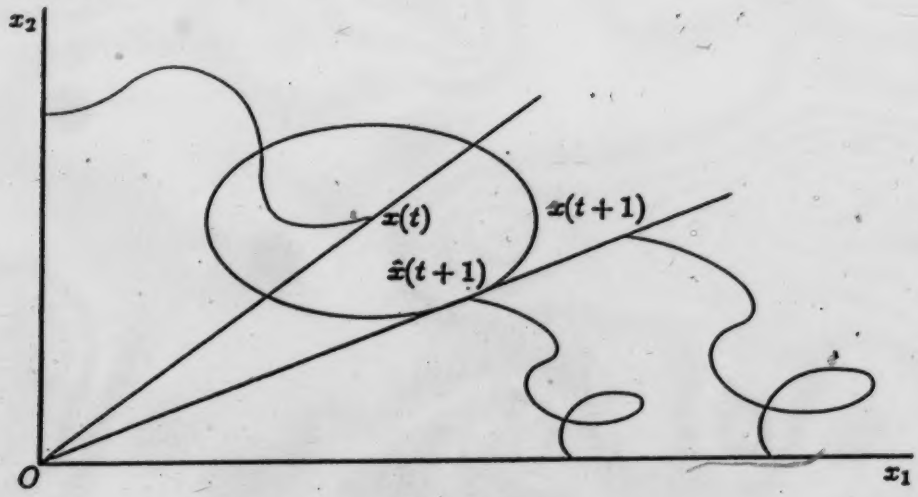

Figure 3.3

We shall say that the polygon x is *locally extremal* with respect to the norm $|\cdot|_2$ if each $x(t)$, $t \in \mathcal{D}_{N-1}$ is the center of a $|\cdot|_2$ – sphere which is supported at $x(t+1)$ by the ray $Ox(t+1)$.

Our discussion gives the following proposition.

PROPOSITION 3.1. *Among all polygons x with vertices on a given ordered set of rays through O, the sum $S_{1,2}x$ is minimized if and only if x is locally extremal.*

A useful geometric realization of the sum $S_{1,2}x$ is as follows:

$$\frac{|x(t+1) - x(t)|_2}{|x(t)|_1} = |Q - P|_2$$

where $P = x(t)/|x(t)|_1, Q = x(t+1)/|x(t)|_1$. Thus $|\Delta\, x(t)|_2/|x(t)|_1$ is the length of a directed line segment with its initial vertex on the sphere $|x|_1 = 1$

which is parallel to the side of the polygon, and $S_{1,2}x$ is the sum of the $|\cdot|_2$ lengths of all such segments.

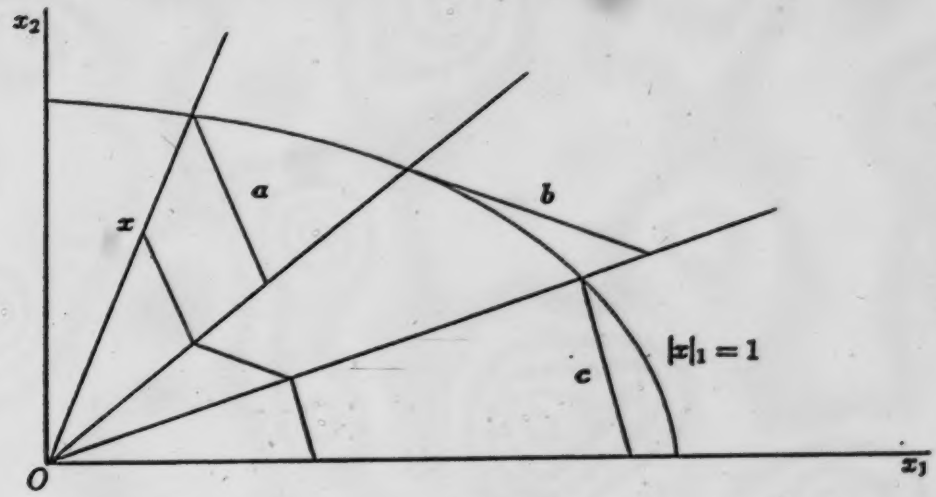

Figure 3.4

For the polygon x in Figure 3.4, $S_{1,2}x = |a|_2 + |b|_2 + |c|_2$. When the polygon x is locally extremal, each of these lengths is the radius of a $|\cdot|_2$–sphere centered on the initial point of the segment and supported by the ray through O and the terminal point.

Recall that the constant $k_{1,2}^{(2)}(N,\delta)$ is the infimum of $S_{1,2}x$ over all polygons in $\mathbb{R}^2$ with N edges which intersect both coordinate axes and satisfy

$$|\Delta x(t)|_2 \leq \delta |x(t)|_1 \quad , \quad t \in \mathcal{D}_{N-1}. \tag{3.5}$$

From Proposition 3.1, it follows that we need only consider locally extremal polygons x. We may also restrict our attention to polygons x such that the edge $[x(0),x(1)]$ intersects one of the coordinate axes, and $[x(N-1),x(N)]$ intersects the other. Moreover, we may assume that $x(N)$ is on the coordinate

axis since $\mathcal{S}_{1,2}x$ is decreased if the last edge is shortened. It may not be assumed *a priori* that $x(0)$ is on the other axis, but this is often the case for the more common norms with δ sufficiently small. In the case that $|\cdot|_1$ and $|\cdot|_2$ are absolute symmetric norms $(|(x_1,x_2)| = |(\pm x_1, \pm x_2)| = |(x_2,x_1)|)$ we may assume that $x \geq 0$ $(x_i(t) \geq 0, i = 1,2$ for $t \in \mathcal{D}_N)$, the edge $[x(0),x(1)]$ intersects the positive x_2-axis, and $x(N)$ is on the positive x_1-axis.

For a given locally extremal polygon x, the issue of whether $x(0)$ may be taken on the coordinate axis can be decided as follows. Suppose that the edge $[x(0),x(1)]$ intersects the positive x_2-axis at $B = (0,\eta)$ as illustrated in Figure 3.6, and $|x(0)|_1 = 1$.

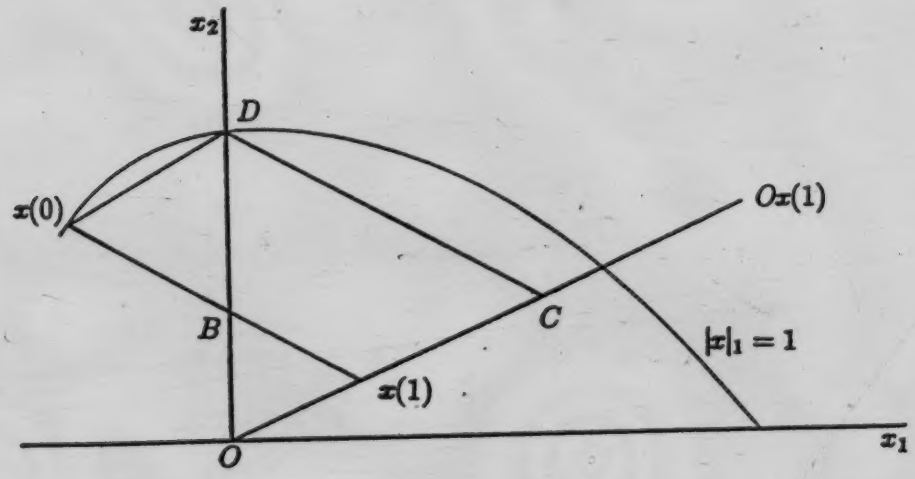

Figure 3.6

Then $\mathcal{S}_{1,2}x$ is decreased by replacing $x(0)$ by B if and only if

$$|x(1) - x(0)|_2 \geq \frac{|x(1) - B|_2}{|B|_1} = |C - D|_2$$

where $D = B/|B|_1$, and $C = x(1)/|B|_1$. This is equivalent to requiring that the slope of the chord $[x(0), D]$ of the sphere $|x|_1 = 1$ is less than the slope of ray $Ox(1)$ as illustrated in Figure 3.6

For example, let $|\cdot|_{p_1}$ and $|\cdot|_{p_2}$ be a pair of Hölder norms. The Hölder norms $|\cdot|_p$, $1 \le p \le \infty$ on $\mathbb{R}^n$ are defined by

$$|x|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \quad \text{for} \quad 1 \le \infty,$$
$$|x|_\infty = \max\{|x_i| : i = 1, \ldots, n\}.$$

The unit spheres in $\mathbb{R}^2$ for $p = 1, 2, \infty$ are as illustrated in Figure 3.7.

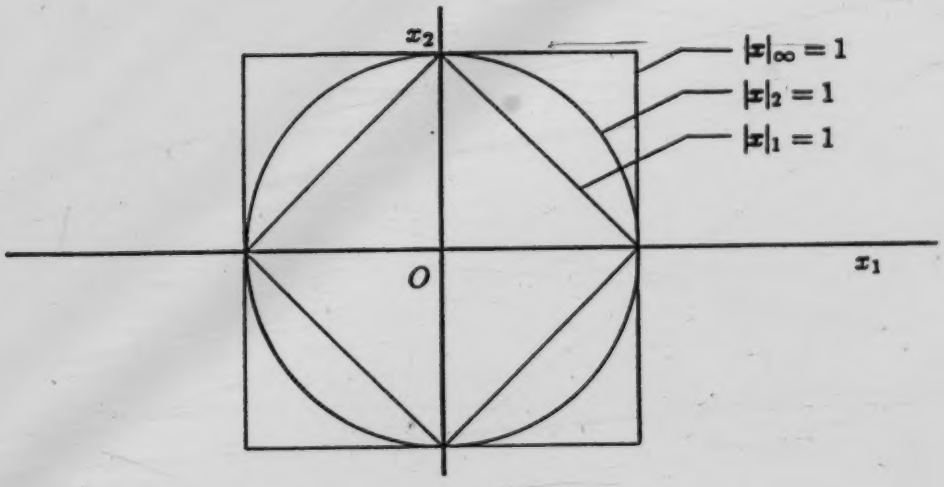

Figure 3.7

Then, if $\delta \le \frac{1}{2}$, $x(0)$ may be taken on the positive $x_2 -$ axis. To see this, observe that in Figure 3.6, with $x(0) = (\zeta_0, \eta_0)$ and $x(1) = (\zeta_1, \eta_1)$,

$$|x(1) - x(0)|_{p_2} \le \frac{1}{2}|x(0)|_{p_1}$$

implies

$$\max\{|\zeta_1 - \zeta_0|, |\eta_1 - \eta_0|\} \leq \frac{1}{2}(|\zeta_0| + |\eta_0|)$$

and therefore

$$(\zeta_1 - \zeta_0) \leq \frac{1}{2}(-\zeta_0 + \eta_0) \quad , \quad (\eta_0 - \eta_1) \leq \frac{1}{2}(-\zeta_0 + \eta_0),$$

so that

$$\zeta_1 \leq \frac{1}{2}(\zeta_0 + \eta_0) \leq \eta_1.$$

Thus the slope η_1/ζ_1 of the ray $Ox(1)$ is at least 1, and since the slope of edge $[x(0), D]$ is at most 1, we may take $x(0) = B$ on the positive x_2-axis.

Another useful situation where $x(0)$ may be taken on the positive x_2-axis is when both norms are absolute symmetric, $|x|_1 \leq |x|_2$ for each $x \in \mathbb{R}^2$, and $0 < \delta \leq 1$. To see this, by Proposition 3.1, it is sufficient to show that the sum $S_{1,2}x$ is decreased if we replace $x(0)$ with $B = x(0) + k\Delta x(0)$ where $0 \leq k < 1$ as illustrated in Figure 3.6. But $|\Delta x(0)|_1 \leq |\Delta x(0)|_2 \leq |x(0)|_1$ implies

$$(1-k)|x(0)|_1 \leq |x(0)|_1 - k|\Delta x(0)|_1 \leq |B|_1.$$

Multiplying this inequality by $|\Delta x(0)|_2$ we find

$$(1-k)|\Delta x(0)|_2|x(0)|_1 \leq |\Delta x(0)|_2|B|_1,$$

which implies

$$|x(1) - B|_2|x(0)|_1 \leq |\Delta x(0)|_2|B|_1$$

since $(1-k)\Delta \boldsymbol{x}(0) = \boldsymbol{x}(1) - B$. Thus

$$\frac{|\boldsymbol{x}(1) - B|_2}{|B|_1} \leq \frac{|\Delta \boldsymbol{x}(0)|_2}{|\boldsymbol{x}(0)|_1}$$

so that the sum $\mathcal{S}_{1,2}\boldsymbol{x}$ is not increased if we replace the edge $[\boldsymbol{x}(0), \boldsymbol{x}(1)]$ by $[B, \boldsymbol{x}(1)]$.

The preceding discussion may be summarized by the following propositions.

PROPOSITION 3.2. *Let* $|\cdot|_{p_1}$ *and* $|\cdot|_{p_2}$ *be Hölder norms and* $0 < \delta \leq \frac{1}{2}$. *Then* $k^{(2)}_{p_1,p_2}$ *is the infimum of* $\mathcal{S}_{p_1,p_2}\boldsymbol{x}$ *over all locally extremal polygons* $\boldsymbol{x} \geq 0$ *with* $\boldsymbol{x}(0)$ *on the positive* x_2 *– axis,* $\boldsymbol{x}(N)$ *on the positive* x_1 *– axis, and* $|\Delta \boldsymbol{x}(t)|_{p_2} \leq \delta |\boldsymbol{x}(t)|_{p_1}$ *for* $t \in \mathcal{D}_{N-1}$.

PROPOSITION 3.3. *Let* $|\cdot|_1$ *and* $|\cdot|_2$ *be absolute symmetric norms with* $|\boldsymbol{x}|_1 \leq |\boldsymbol{x}|_2$ *for* $\boldsymbol{x} \in \mathbb{R}^2$, *and* $0 < \delta \leq 1$. *Then* $k^{(2)}_{1,2}$ *is the infimum of* $\mathcal{S}_{1,2}\boldsymbol{x}$ *over all locally extremal polygons* $\boldsymbol{x} \geq 0$ *with* $\boldsymbol{x}(0)$ *on the positive* x_2 *– axis,* $\boldsymbol{x}(N)$ *on the positive* x_1 *– axis, and* $|\Delta\, \boldsymbol{x}(t)|_2 \leq \delta |\boldsymbol{x}(t)|_1$ *for* $t \in \mathcal{D}_{N-1}$.

§3.2 The Constant $k^{(2)}_{p,\infty}(N, \delta)$

We now consider the problem of determining the constant $k_{p,\infty}$ where $|\cdot|_p$ for $1 \leq p \leq \infty$, is the Hölder norm. Because the $|\cdot|_\infty$ – spheres are supported by rays through the origin (except the axes) on their corners, the locally extremal polygons with respect to $|\cdot|_\infty$ joining the positive coordinate axes together consist entirely of edges with slope -1. This is illustrated in Figure 3.8.

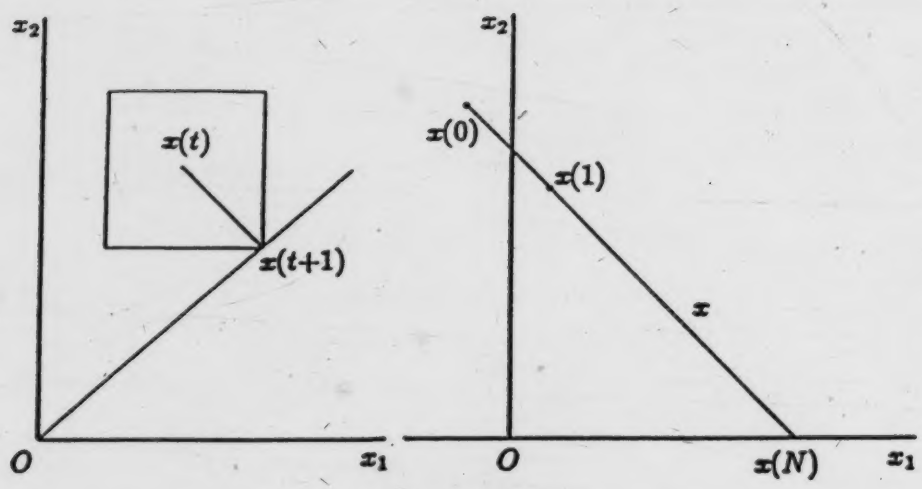

Figure 3.8

The situation where $p = 1$ is particularly simple. First, suppose that $\delta \leq \frac{1}{2}$; then we may take $x(0) = (0,1)$ so that every locally extremal polygon x lies on the sphere $|x|_1 = 1$. In this case we have

$$S_{1,\infty}x = \sum_{0}^{N-1} |\zeta_{t+1} - \zeta_t| \geq 1$$

where $x(t) = (\zeta_t, 1-\zeta_t), 0 \leq \zeta_t \leq 1$, and $S_{1,\infty}x = 1$ when $\zeta_t \leq \zeta_{t+1}$. Next, suppose that $\frac{1}{2} < \delta \leq 1$; then we cannot assume *a priori* that $x(0) = (0,1)$. Let $x(0) = (-\zeta_0, 1+\zeta_0)$ and observe that

$$S_{1,\infty}x \geq \frac{\zeta_1 + \zeta_0}{1+2\zeta_0} + |\sum_{1}^{N-1}(\zeta_{t+1} - \zeta_t)| \geq \frac{1}{2}.$$

Define x by $x(0) = (-\zeta, 1+\zeta)$, and $x(t) = (1,0)$ for $t = 1, \ldots, N$. Then, for sufficiently large ζ, x satisfies $|\Delta x(t)|_\infty \leq \delta |x(t)|_1$ for $t \in \mathcal{D}_{N-1}$, and

$$S_{1,\infty}x = \frac{\zeta+1}{1+2\zeta} \to \frac{1}{2} \quad as \quad \zeta \to +\infty.$$

Thus we have the following theorem.

THEOREM 3.4. *Suppose* $|\cdot|_1$ *and* $|\cdot|_\infty$ *are the Hölder norms for* $p = 1, \infty$. *Then*

$$k_{1,2}^{(2)}(N,\delta) = \begin{cases} 1 & \text{if } 0 < \delta \le \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} < \delta \le 1 \,. \end{cases}$$

To determine $k_{p,\infty}^{(2)}$ for $1 < p < \infty$ is difficult in general. However, the corresponding constant $\hat{c}_{p,\infty}^{(2)}$ can be determined. By our discussion thus far we know that for $\delta \le \frac{1}{2}$, $k_{p,\infty}^{(2)}$ is the infimum of

$$\sum_0^{N-1} \frac{|\Delta\zeta_t|}{|(\zeta_t, 1-\zeta_t)|_p} \tag{3.9}$$

subject to $0 = \zeta_0 \le \zeta_t \le \zeta_N = 1$, and $|\Delta\zeta_t| \le \delta|(\zeta_t, 1-\zeta_t)|_p$. Because

$$\frac{|\zeta_{t+2} - \zeta_t|}{|(\zeta_t, 1-\zeta_t)|_p} \le \sum_{s=t,t+1} \frac{|\Delta\zeta_t|}{|(\zeta_t, 1-\zeta_t)|_p}$$

where $\zeta_{t+1} \le \zeta_t, \zeta_{t+2}$, we may also assume that $\zeta_t \le \zeta_{t+1}$. In this case the sum in (3.9) becomes

$$S(\mathcal{P}_N, f) = \sum_0^{N-1} f(\zeta_t)\Delta\zeta_t$$

which is a Riemann sum of $f(\zeta) = |(\zeta, 1-\zeta)|_p^{-1}$ with partition $\mathcal{P}_N = \{\zeta_t : t \in \mathcal{D}_N\}$. If $\delta \to 0^+$, then $|\mathcal{P}_{N(\delta)}| \to 0$ where $|\mathcal{P}_N| = \max\{|\Delta\zeta_t| : t \in \mathcal{D}_{N-1}\}$ so that $S(\mathcal{P}_{N(\delta)}, f)$ approaches the integral

$$\int_0^1 \frac{d\zeta}{|(\zeta, 1-\zeta)|_p} \,.$$

Making the change of variable $\zeta = (1+s)^{-1}$ shows that the limit of $k_{p,\infty}^{(2)}$ as $\delta \to 0^+$ is

$$c_{p,\infty}^{(2)} = 2\int_0^1 \frac{ds}{|(1,s)|_p |(1,s)|_1} \,.$$

Thus we have the following theorem (see [1,p.47]).

THEOREM 3.5. *Suppose* $|\cdot|_p$ *is the Hölder norm for* $1 \le p \le \infty$. *Then*

$$\hat{c}^{(2)}_{p,\infty} = c^{(2)}_{p,\infty}$$

where $c^{(2)}_{p,\infty}$ *is the corresponding nonoscillation constant for differential systems.*

Before we consider the constant $k^{(2)}_{\infty,\infty}$, we prove the preliminary lemmas.

LEMMA 3.6. *Suppose that* $0 < \delta \le 1$ *and* $x, y,$ *and* z *satisfy the conditions*

(i) $0 \le x \le y \le z \le \frac{1}{2}$,

(ii) $(y - x) \le \delta(1 - x), \quad (z - y) \le \delta(1 - y)$

Then the function

$$f(y) = \frac{y - x}{1 - x} + \frac{z - y}{1 - y}$$

is minimized at $y = z$ *if* $(z - x) \le \delta(1 - x)$, *and at* $y = \delta(1 - x) + x$ *otherwise.*

PROOF: If $(z - x) \le \delta(1 - x)$, then $x \le y$ implies

$$\frac{z - x + x - y}{1 - x} \le \frac{z - y}{1 - y},$$

and therefore

$$f(z) = \frac{z - x}{1 - x} \le \frac{y - x}{1 - x} + \frac{z - y}{1 - y} = f(y).$$

However, if $(z - x) > \delta(1 - x)$, then from condition (ii) it suffices to show $f(y)$ is minimized on the boundary of $[(z - \delta)/(1 - \delta), x - \delta(1 - x)]$. But this is true because $f'(y) = 0$ implies $f''(y) < 0$, and therefore f has a relative maximum when $f'(y) = 0$.

□

LEMMA 3.7. *Suppose that* $0 < \delta \leq 1$, *and* x, y, z *satisfy*

(i) $\frac{1}{2} \leq x \leq y \leq z \leq 1$,

(ii) $(y - x) \leq \delta x, \quad (z - y) \leq \delta y$

Then the function

$$f(y) = \frac{y - x)}{x} + \frac{z - y}{y}$$

is minimized when $y = \sqrt{xz}$.

LEMMA 3.8. *Suppose that*

(i) $0 < \delta \leq 1 \quad , \quad m \in \mathbb{N}_{>0}$,

(ii) $n = \sup\{k \in \mathbb{N} : 1 < 2(1 - \delta)^{k-1}\}$,

(iii) $B = 1 - (1 - \delta)^{n-1}$.

Then the function

$$f(x) = \frac{x - B}{1 - B} + m(x^{-\frac{1}{m}} - 1)$$

has minimum at $x = \min\{(1 - B)^{\frac{m}{m+1}}, 1 - (1 - \delta)^n\}$ *on the closed interval* $[\frac{1}{2}, 1 - (1 - \delta)^n]$.

PROOF: : If $f'(x) = 0$, then $x = (1-B)^{\frac{m}{m+1}} = (1-\delta)^{\frac{m(n-1)}{m+1}}$, and $f''(x) < 0$. So it suffices to show $(1 - B)^{\frac{m}{m+1}} \geq \frac{1}{2}$. But condition (ii) implies

$$\frac{1}{2} < (1 - \delta)^{n-1},$$

so that

$$\frac{1}{2} < (\frac{1}{2})^{\frac{m}{m+1}} < (1 - \delta)^{\frac{m(n-1)}{m+1}} = (1 - B)^{\frac{m}{m+1}}.$$

□

Let $0 < \delta \leq 1$. If we take $p = \infty$ in (3.9), then the constant $k^{(2)}_{\infty,\infty}$ is the infimum of the sum

$$\mathcal{S}_{\infty,\infty}\boldsymbol{x} = \sum_{0}^{n-1} \frac{\Delta\zeta_t}{1-\zeta_t} + \sum_{n}^{N-1} \frac{\Delta\zeta_t}{\zeta_t} \tag{3.10}$$

where $0 = \zeta_0 \leq \zeta_t \leq \zeta_{t+1} \leq \zeta_N = 1$ satisfy $\zeta_t \leq 1/2$ for $t \in \mathcal{D}_{n-1}$; $\zeta_t \geq 1/2$ for $t \in \mathcal{D}_{N-1}\backslash\mathcal{D}_{n-1}$; and $\Delta\zeta_t \leq \delta|(\zeta_t, 1-\zeta_t)|_\infty$ for $t \in \mathcal{D}_{N-1}$. If we define

$$n := \sup\{k \in \mathbb{N} : \frac{1}{2} < (1-\delta)^{k-1}\},$$

then, by Lemma 3.6, $\mathcal{S}_{\infty,\infty}\boldsymbol{x}$ in (3.10) is minimized only if $\zeta_t = 1-(1-\delta)^t$ for $t \in \mathcal{D}_{n-1}$. This is equivalent to taking $\boldsymbol{x}$ so that $|\Delta\, \boldsymbol{x}(t)| = \delta|\boldsymbol{x}(t)|$ for $t \in \mathcal{D}_{n-2}$. In this case $\mathcal{S}_{\infty,\infty}\boldsymbol{x}$ in (3.10) becomes

$$(n-1)\delta + \frac{\Delta\zeta_{n-1}}{1-\zeta_{n-1}} + \sum_{n}^{N-1} \frac{\Delta\zeta_t}{\zeta_t} \tag{3.11}$$

where $\zeta_t \geq \frac{1}{2}$ for $t \in \mathcal{D}_N\backslash\mathcal{D}_{n-1}$. Lemma 3.7 implies that the sum in (3.11) is minimized only if $\zeta_{t+1} = \sqrt{\zeta_t\zeta_{t+2}}$ for $t \in \mathcal{D}_N\backslash\mathcal{D}_n$. So (3.11) becomes

$$(n-1)\delta + \frac{\Delta\zeta_{n-1}}{1-\zeta_{n-1}} + m(\zeta_n^{-\frac{1}{m}} - 1)$$

where $m+n = N$. But minimizing this expression as a function of ζ_n subject to the conditions $\Delta\zeta_{n-1} \leq \delta(1-\zeta_{n-1})$, and $\zeta_n \geq \frac{1}{2}$ implies that $\zeta_n = \min\{(1-\delta)^{\frac{(n-1)m}{m+1}}, 1-(1-\delta)^n\}$ by Lemma 3.8.

Thus we have the theorem.

THEOREM 3.9. *Suppose that* $|\cdot|_\infty$ *is the* $p=\infty$ *Hölder norm and* $0<\delta\leq 1$. *Then*

$$k^{(2)}_{\infty,\infty}(N,\delta)=F(\zeta)$$

where

(i) $n=\sup\{k\in\mathbb{N}: 1<2(1-\delta)^{k-1}\}$,

(ii) $N=m+n$,

(iii) $B=1-(1-\delta)^{n-1}$,

(iv) $\zeta=\min\{(1-\delta)^{\frac{m(n-1)}{m+1}}, 1-(1-\delta)^n\}$,

(v) $F(x)=(n-1)\delta+\frac{x-B}{1-B}+m(x^{-\frac{1}{m}}-1)$.

§3.3 The Constant $k^{(2)}_{p,1}(N,\delta)$

Before we consider the constant $k^{(2)}_{p,1}$ where $|\cdot|_p$ is the Hölder norm, we first state and prove some preliminary results.

LEMMA 3.10. *Let* $0<\delta<1$. *Then the function*

$$f(x,y)=1-y+x+\frac{y}{x}$$

defined on $S=\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq 1, y\geq 1-\delta\}$ *is minimized at* $(x,y)=\left((1-\delta)^{\frac{1}{2}},1-\delta\right)$.

PROOF: Since $f_y(x,y)=-1+x^{-1}\geq 0$ on S, f is minimized only if $y=1-\delta$. Therefore f is minimized on S when $f(x,1-\delta)$ is minimized on $[0,1]$. To finish the proof, observe that $f'(x,1-\delta)=0$ implies $x=(1-\delta)^{\frac{1}{2}}$ and $f''(x,1-\delta)>0$.

□

LEMMA 3.11. *Suppose* $0 < \delta < 1$, *and* x, y, z *and* a *satisfy*

(i) $0 \le x \le y \le z \le a \le 1$,

(ii) $(z - y) \le \delta(a + z)$, $(y - x) \le \delta(a + y)$.

Then the function

$$f(y) = \frac{z - y}{a + z} + \frac{y - x}{a + y}$$

is minimized at $y = \max\{x, z - \delta(a + z)\}$.

PROOF: If $(z - x) \le \delta(a + z)$, then $x \ge z - \delta(a + z)$. So we have to show $f(x) \ge f(y)$. But $y \le z$ implies

$$\frac{z - z - x + y}{a + z} \le \frac{y - x}{a + y}$$

and therefore

$$f(x) = \frac{z - x}{a + z} \le \frac{z - y}{a + z} + \frac{y - x}{a + y} = f(y).$$

If $(z - x) > \delta(a + z)$ then minimizing f is equivalent to minimizing f on $[z - \delta(a + z), (x + a\delta)/(1 - \delta)]$. But $f'(y) = 0$ implies $f''(y) < 0$. Thus f is minimized on the boundary of its domain.

□

LEMMA 3.12. *Suppose that*

$$f(x) = \frac{x - a}{1 + a} + \frac{b - x}{1 + x}$$

is defined on $[a, b]$. *Then* f *is minimized only if* $x = \sqrt{(1 + a)(1 + b)} - 1$.

LEMMA 3.13. *Suppose that*

(i) $0 < \delta < 1, \quad m \in \mathbb{N} > 0,$

(ii) $n = \sup\{k \in \mathbb{N} : 1 < 2(1-\delta)^{k-1}\},$

(iii) $A = [1-(1-\delta)^{n-1}]/(1-\delta)^{n-1}.$

Then

$$E(x) = \frac{1-xA}{1+x} + m[(1+x)^{\frac{1}{m}} - 1]$$

has infimum at $x = \max\{(1+A)^{\frac{m}{m+1}} - 1,\ (1-\delta)/(1+A)\}$ *on the domain* $[(1-\delta)/(1+A), 1]$.

PROOF: If $E'(x) = 0,$ then $x = (1+A)^{\frac{m}{m+1}} - 1,$ and $E''(x) > 0.$ So to finish the proof we need to show $(1+A)^{\frac{m}{m+1}} - 1 \leq 1,$ but conditions (ii) and (iii) imply

$$\frac{1}{2} < (\frac{1}{2})^{\frac{m}{m+1}} \leq (1+A)^{\frac{-m}{m+1}},$$

and therefore $(1+A)^{\frac{m}{m+1}} - 1 \leq 1.$

□

To determine $k^{(2)}_{p,1},$ the infimum of $\mathcal{S}_{p,1}\boldsymbol{x}$ over all locally extremal polygons $\boldsymbol{x}$ joining the positive x_2-axis to the positive x_1-axis satisfying the condition $|\Delta\boldsymbol{x}|_1 \leq \delta|\boldsymbol{x}|_p$ on $\mathcal{D}_{N-1}$ must be calculated. Because the $|\cdot|_1$-spheres are supported by rays through the origin on their corners, the locally extremal polygons satisfy (as illustrated in Figure 3.12a) these two conditions

(i) If the ray $O\boldsymbol{x}(t+1)$ has slope greater than 1, then the edge $[\boldsymbol{x}(t), \boldsymbol{x}(t+1)]$ has zero slope.

(ii) If the ray $Ox(t+1)$ has slope less than 1, then the edge $[x(t), x(t+1)]$ is parallel to the x_2-axis.

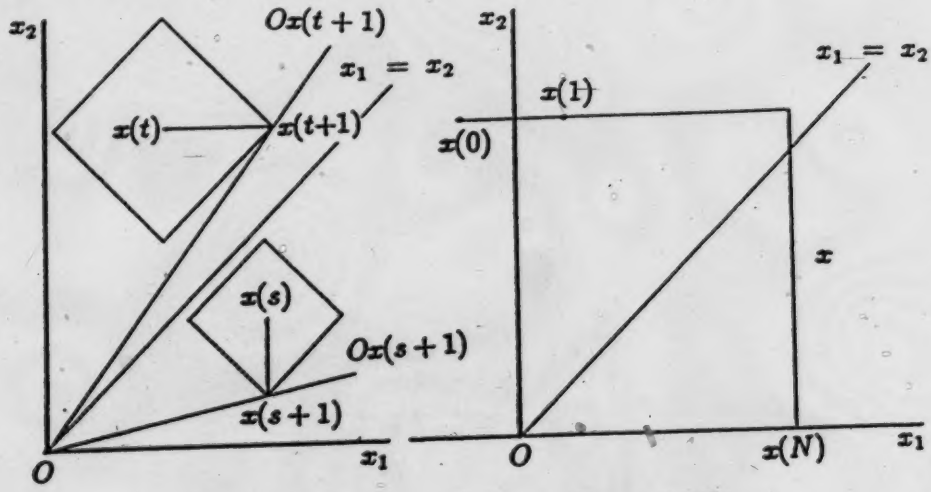

Figure 3.12a

Figure 3.12b

It is not difficult to see that we need only consider those polygons x such that the slope of $Ox(t)$ decreases as t increases. Such locally extremal polygons are illustrated in Figure 3.12b, and in this case we may write

$$S_{p,1}x = \sum_{t<n} \frac{\Delta\xi_t}{(1+\xi_t^p)^{1/p}} + \sum_{t>n} \frac{|\Delta\zeta_t|}{(\xi_n^p+\zeta_t^p)^{1/p}} + \frac{1-\zeta_{n+1}}{(1-\xi_n^p)^{1/p}} \tag{3.13}$$

where $x(t) = (\xi_t, 1)$ for $t \le n$, and $x(t) = (\xi_n, \zeta_t)$ for $t > n$ with $-1 < \xi_0 \le 0 \le \xi_t \le \xi_{t+1} \le 1$ and $0 \le \zeta_{t+1} \le \zeta_t \le \xi_n$.

The situation where $p = \infty$ is quite simple. In this case we may assume $\xi_0 = 0$ so that $S_{\infty,1}x$ in (3.13) becomes

$$S_{\infty,1}x = 1 - \zeta_{n+1} + \xi_n + \zeta_{n+1}\xi_n^{-1}. \tag{3.14}$$

Since we require $N \ge 2(\delta+1)/\delta$ by Theorem 2.1, and $|x(t)|_1 \le \delta|x(t)|_\infty$ for $t \in \mathcal{D}_{N-1}$, ξ_n and ζ_{n+1} must satisfy the conditions $0 \le \xi_n \le 1$ and

$1-\zeta_{n+1} \le \delta$. But by Lemma 3.10, the minimum of $\mathcal{S}_{\infty,1}x$ in (3.14) subject to $0 \le \xi_n \le 1$ and $1-\zeta_{n+1} \le \delta$ occurs when $(\xi_n, \zeta_{n+1}) = ((1-\delta)^{\frac{1}{2}}, 1-\delta)$.

Thus we have the following theorem.

THEOREM 3.14. *Suppose that* $|\cdot|_1$ *and* $|\cdot|_\infty$ *are Hölder norms for* $p = 1, \infty$, *and* $0 < \delta \le 1$. *Then*

$$k^{(2)}_{\infty,1}(N,\delta) = \delta + 2(1-\delta)^{\frac{1}{2}} .$$

To determine $k^{(2)}_{p,1}$ for $1 < p < \infty$ is difficult in general. However, the corresponding constant $\hat{c}^{(2)}_{p,1}$ can be determined. If $\delta \le 1/2$, then for $\mathcal{S}_{p,1}x$ in (3.13) we may take $\xi_0 = 0$. Let $\delta \to 0^+$. Then the condition $|\Delta x(t)|_1 \le \delta |x(t)|_p$ implies that $\sup \Delta\xi_t \to 0^+$, and $\sup \Delta\zeta_t \to 0^+$ in $\mathcal{S}_{p,1}x$, and therefore the limit as $\delta \to 0^+$ of $k^{(2)}_{p,1}(N(\delta),\delta)$ is

$$c^{(2)}_{p,1} = 2\int_0^1 \frac{ds}{|(1,s)|_p} .$$

Thus we have the following theorem (see [1,p.47]).

THEOREM 3.15. *Suppose that* $|\cdot|_p$ *and* $|\cdot|_\infty$ *are Hölder norms. Then*

$$\hat{c}^{(2)}_{p,1} = c^{(2)}_{p,1}$$

where $c^{(2)}_{p,1}$ *is the corresponding nonoscillation constant for differential systems.*

To determine $k^{(2)}_{1,1}$ for $0 < \delta < 1$, the minimum of $\mathcal{S}_{1,1}x$ in (3.13) or

$$\mathcal{S}_{1,1}x = \sum_{t<m} \frac{\Delta\xi_t}{(1+\xi_t)} + \sum_{t>m} \frac{|\Delta\zeta_t|}{(\xi_m+\zeta_t)} + \frac{1-\xi_{m+1}}{1+\xi_m} \qquad (3.15)$$

subject to $\Delta\xi_t \le \delta(1+\xi_t)$, $|\Delta\zeta_t| \le \delta(\xi_m+\zeta_t)$, and $1-\zeta_{m+1} \le \delta(1+\xi_m)$ must be calculated where $0=\xi_0 \le \xi_t \le \xi_{t+1} \le 1$, and $0 \le \zeta_{t+1} \le \zeta_t \le \xi_m$.

If we define

$$n := \sup\{k \in \mathbb{N} : \ 1 < 2(1-\delta)^{k-1}\},$$

then by Lemma 3.11, the sum

$$\sum_{t>m} \frac{\Delta\zeta_t}{(\xi_m+\zeta_t)}$$

where $m = N-n$ is minimized only if

$$\zeta_{N-t} = \xi_m \frac{[1-(1-\delta)^t]}{(1-\delta)^t}$$

for $t \in \mathcal{D}_{n-1}$. This is equivalent to taking $|\Delta x(t)|_1 = \delta|x(t)|_1$ for $t \in \mathcal{D}_N/\mathcal{D}_m$. In this case $\mathcal{S}_{1,1}x$ in (3.15) becomes

$$\mathcal{S}_{1,1}x = (n-1)\delta + \frac{1-\zeta_{m+1}}{1+\xi_m} + \sum_{t<m} \frac{\Delta\xi_t}{1+\xi_t}.$$

From Lemma 3.12, this sum is minimized only if

$$\xi_t = \sqrt{(1+\xi_{t-1})(1+\xi_{t+1})} - 1$$

for $t \in \mathcal{D}_{m-1}$, and therefore the sum becomes

$$\mathcal{S}_{1,1}x = (n-1)\delta + \frac{1-\xi_m A}{1+\xi_m} + m[(1+\xi_m)^{\frac{1}{m}} - 1] \tag{3.16}$$

where $A = [1-(1-\delta)^{n-1}]/(1-\delta)^{n-1}$. We need to minimize $\mathcal{S}_{1,1}x$ in (3.16) subject to $1-\xi_m A \le \delta(1+\xi_m)$ and $0 \le \xi_m \le 1$ since we require $|\Delta x|_1 \le \delta|x|_1$ on $\mathcal{D}_N$. But Lemma 3.13 implies that this occurs when $\mathcal{S}_{1,1}x$ in (3.16) is evaluated at $\xi_n = \max\{(1+A)^{\frac{m}{m+1}} - 1, (1-\delta)/(1+A)\}$.

Thus we have the following theorem.

THEOREM 3.16. *Let* $|\cdot|_1$ *be the Hölder norm for* $p = 1$, *and* $0 < \delta < 1$. *Then*

$$k_{1,1}^{(2)}(N, \delta) = E(s).$$

where

(i) $n = \sup\{k \in \mathbb{N} : 1 < 2(1-\delta)^{k-1}\}$,

(ii) $N = m + n$,

(iii) $A = [1 - (1-\delta)^{n-1}]/(1-\delta)^{n-1}$,

(iv) $s = \max\{(1+A)^{\frac{m}{m+1}} - 1, (1-\delta)/(1+A)\}$,

(v) $E(x) = (n-1)\delta + (1 - xA)/(1+x) + m[(1+x)^{\frac{1}{m}} - 1]$.

Moreover, if $s = (1-\delta)/(1+A)$, *then*

$$k_{1,1}^{(2)}(N, \delta) = n\delta + m\{[1 - (1-\delta)^n]^{-\frac{1}{m}} - 1\};$$

and if $n = m$ *and* $\delta = 1 - (\frac{1}{2})^{\frac{1}{n}}$, *then*

$$k_{1,1}^{(2)}(N, \delta) = 2n \sinh(\frac{ln2}{n}).$$

§3.4 The Constant $k_{2,2}^{(2)}(N, \delta)$

In this section we consider the constant $k = k_{2,2}^{(2)}(N, \delta)$ where $|\cdot| = |\cdot|_2$ is the Euclidean norm.

THEOREM 3.17. *Let* $\theta = [\pi/(2\sin^{-1}\delta)]$ *where* $[\,\cdot\,]$ *is the greatest integer function. Then*

$$k(N, \delta) = \theta\delta + \sin(\frac{\pi}{2} - \theta\delta).$$

Moreover,

$$\hat{c} = \frac{\pi}{2}$$

which is the corresponding nonoscillation constant for differential systems.

PROOF: From Proposition 3.3 we need to determine the infimum of $S_{2,2}x$ over those polygons x extending from the positive x_2-axis to the positive x_1-axis such that $x(t) \geq 0$ and $|\Delta x(t)| \leq |x(t)|$ for $t \in \mathcal{D}_{N-1}$.

Let $t \in \mathcal{D}_{N-1}$. Then the edge $[x(t), x(t+1)]$ is perpendicular to the ray $Ox(t+1)$ because the $|\cdot|-$sphere centered on $x(t)$ and supported by $Ox(t+1)$ at $x(t+1)$ is a circle. This is illustrated in Figure 3.17.

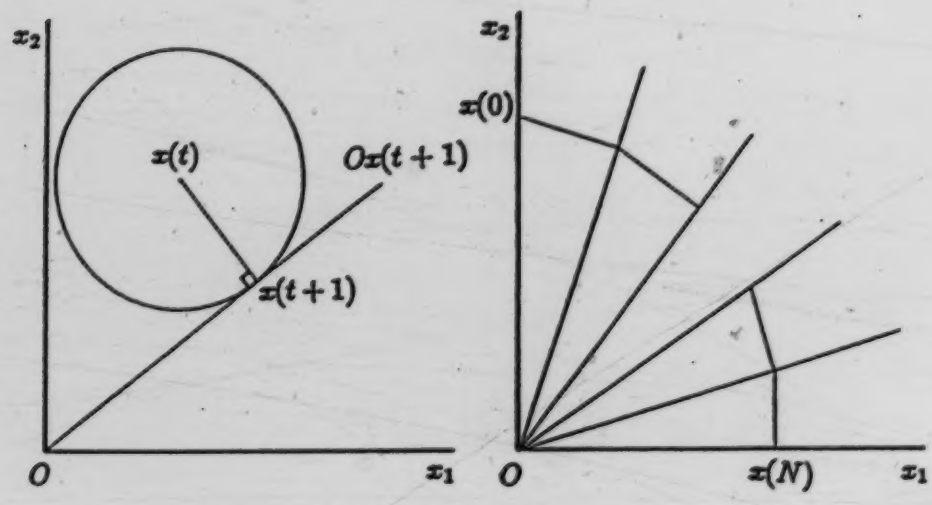

Figure 3.17

When considering the quotient $|\Delta x(t)|/|x(t)|$, we may assume $|x(t)| = 1$. So since the edges $[x(t), x(t+1)]$ and $[O, x(t+1)]$ are at a right angle to each other,

$$\frac{|\Delta x(t)|}{|x(t)|} = \sin\theta_t$$

where $\theta_t \geq 0$ is the angle between $x(t)$ and $x(t+1)$. This implies that k is the infimum of

$$\mathcal{S}_{2,2}x = \sum_0^{N-1} \sin\theta_t$$

where $\sin\theta_t \leq \delta$, and $\sum_0^{N-1} \theta_t = \frac{\pi}{2}$. It follows from Lemma 3.18 below that to minimize $\mathcal{S}_{2,2}x$, the angle θ_t should be chosen as large as is permitted by the condition

$$\sin\theta_t \leq \frac{|\Delta x(t)|}{|x(t)|} \leq \delta.$$

Therefore we should have $\sin\theta_t = \delta$ whenever possible and consequently

$$k = \theta\delta + \sin(\frac{\pi}{2} - \theta\delta)$$

where $\theta = [\pi/(2\sin^{-1}\delta)]$ and $[\,\cdot\,]$ is the greatest integer function.

□

LEMMA 3.18. *Suppose that*

(i) c *and* δ *are fixed such that* $0 < c \leq \pi/2$ *and* $0 < \delta \leq 1$,

(ii) $\theta\,,\ \phi \geq 0$, *and* $\theta + \phi = c$,

(iii) $\sin\theta,\ \sin\phi \leq \delta$.

Then

$$\sin\theta + \sin\phi \geq \begin{cases} \sin c & \text{if} \quad \sin\theta + \sin\phi \leq \delta, \\ \delta + \sin(c - \sin^{-1}\delta) & \text{otherwise.} \end{cases}$$

PROOF: If $\sin\theta + \sin\phi \leq \delta$, then the inequality $\sin c \leq \sin\theta + \sin\phi$ follows from the identity $\sin(\theta+\phi) = \sin\theta\cos\theta + \sin\phi\cos\phi$. If $\sin\theta + \sin\phi > \delta$, then define $g(\theta,\phi) = \sin\theta + \sin\phi$. In this case

$$g(\theta,\phi) = 2\cos(\frac{\theta-\phi}{2})\sin(\frac{c}{2})\,.$$

Since g is a decreasing function of $|\theta-\phi|$ for $0 \le |\theta-\phi| \le \frac{\pi}{2}$, it obtains its minimum when $|\theta-\phi|$ is largest. This occurs if θ and ϕ are made respectively as large and as small as is permitted. From condition (iii), θ is largest when $\sin\theta = \delta$. Thus

$$g(\sin^{-1}\delta, c-\sin^{-1}\delta) \le g(\theta,\phi)\ ,$$

or

$$\delta + \sin(c-\sin^{-1}\delta) \le \sin\theta + \sin\phi\ .$$

□

§3.5 The Constant $k_{1,2}^{(2)}(N,\delta)$

We devote this section to the constant $k_{1,2}^{(2)}$ where $|\cdot|_1, |\cdot|_2$ are a pair of absolute symmetric norms.

Let $x(t) = (x_1(t), x_2(t))$ be a locally extremal polygon in the first quadrant ($x \ge 0$). If the angle between $x(t)$ and the x_2-axis is θ_t, then we show

$$\frac{|\Delta x(t)|_2}{|x(t)|_1} = \frac{|\Delta\tan\theta_t|}{|(\tan\theta_t,1)|_1|(1,-\tan\theta_{t+1})|_2^*} \tag{3.18}$$

where $|\cdot|^*$ is the conjugate norm of $|\cdot|$. Let (a_1,a_2) be a linear functional defined on $\mathbb{R}^2$. From the definition of $|\cdot|^*$, if $(b_1,b_2)\in\mathbb{R}^2$,

$$|\sum_{i=1,2} a_i b_i| \le |(a_1,a_2)|_2^*|(b_1,b_2)|_2\ . \tag{3.19}$$

In addition, it may be shown that if $|\cdot|_2$ is UF, then the equality sign in (3.19) holds if and only if

$$a_i = |(a_1,a_2)|_2^* \tfrac{\partial}{\partial x_i}|(b_1,b_2)|_2 \quad \text{for} \quad i=1,2. \tag{3.20}$$

Assume $|\cdot|_2$ is UF. Then (3.19) implies

$$|x_2(t+1)\Delta x_1(t) - x_1(t+1)\Delta x_2(t)| \leq |\Delta x(t)|_2 |(x_2(t+1), -x_1(t+1))|_2^*$$

or

$$|\Delta(\frac{x_1(t)}{x_2(t)})| \leq \frac{|\Delta x(t)|_2 |(x_2(t+1), -x_1(t+1))|_2^*}{|x_2(t)x_2(t+1)|}$$

and therefore

$$\frac{|\Delta x(t)|_2}{|x(t)|_1} \geq \frac{|\Delta \tan\theta_t|}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*} . \tag{3.21}$$

Because x is locally extremal, the ray $Ox(t+1)$ supports the $|\cdot|_2$ – sphere centered on $x(t)$ with radius $|\Delta x(t)|_2$ at $x(t+1)$. This implies that the vector grad $|\Delta x(t)|_2$ is orthogonal to $x(t+1)$, or

$$x(t+1) = k(-\tfrac{\partial}{\partial x_2}|\Delta x(t)|_2, \tfrac{\partial}{\partial x_1}|\Delta x(t)|_2)$$

for some $k > 0$ as illustrated in Figure 3.22.

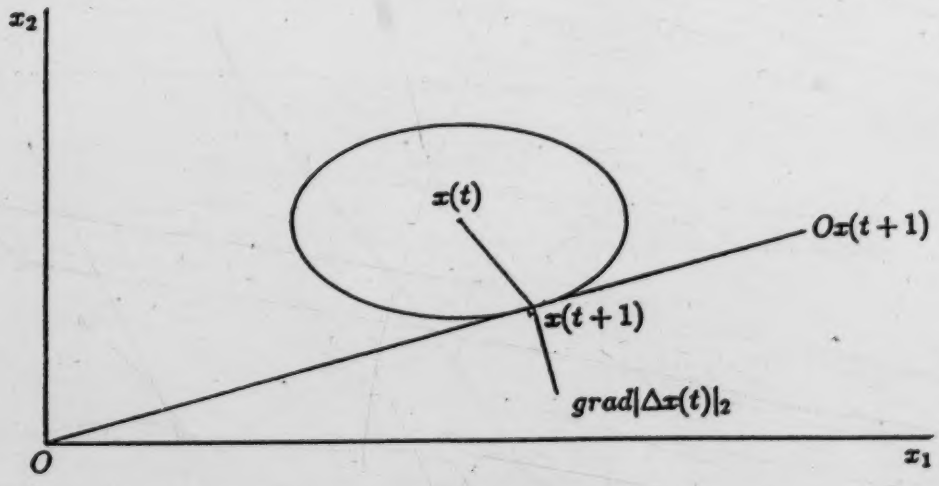

Figure 3.22

To show that the equality sign in (3.21) holds, let

$$y = |y|_2^*(-\tfrac{\partial}{\partial x_2}|\Delta x(t)|_2, \tfrac{\partial}{\partial x_1}|\Delta x(t)|_2) .$$

Then $Oy = Ox(t+1)$, and therefore (3.21) still holds with $x(t+1)$ replaced by y. So by (3.20) the vector y satisfies the equality

$$\frac{|x(t)-y|_2}{|x(t)|_1} = \frac{|\Delta \tan\theta_t|}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*} .$$

But this together with the inequality in (3.21) implies

$$|\Delta x(t)|_2 \geq |x(t)-y|_2 ,$$

and therefore $y = x(t+1)$ because x is locally extremal with respect to the rays $Ox(t)$, and $Ox(t+1) = Oy$. Thus (3.18) holds if $|\cdot|_2$ is UF. In the case that $|\cdot|_2$ is not UF, from Friedland [1,p.6], given any $\epsilon > 0$, there exists a UF norm $|\cdot|_3$ such that

$$|x|_3(1-\epsilon) \leq |x|_2 \leq |x|_3(1+\epsilon),$$

$$|x|_3^*(1-\epsilon) \leq |x|_2^* \leq |x|_3^*(1+\epsilon).$$

Let $\epsilon > 0$ and $|\cdot|_3$ be as above. Then

$$\frac{|\Delta x(t)|_3}{|x(t)|_1} = \frac{|\Delta \tan\theta_t|}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_3^*}$$

implies

$$\frac{|\Delta \tan\theta_t|(1-\epsilon)^2}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*} \leq \frac{|\Delta x(t)|_2}{|x(t)|_1} \leq \frac{|\Delta \tan\theta_t|(1+\epsilon)^2}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*} .$$

So since $\epsilon > 0$ is arbitrary, (3.18) holds. We now summarize this discussion in the following proposition.

PROPOSITION 3.19. *Suppose that* $x \geq 0$ *is a locally extremal polygon extending from the positive* $x_2 -$ *axis to the positive* $x_1 -$ *axis. If* θ_t *is the angle between the positive* $x_2 -$ *axis and* $x(t)$, *then*

$$\frac{|\Delta x(t)|_2}{|x(t)|_1} = \frac{|\Delta \tan \theta_t|}{|(\tan \theta_t, 1)|_1 |(1, -\tan \theta_{t+1})|_2^*}$$

where $|\cdot|^*$ *denotes the conjugate norm of* $|\cdot|$.

From Propositions 3.1 and 3.19 we have the following theorem.

THEOREM 3.20. *Suppose that* $|\cdot|_1$ *and* $|\cdot|_2$ *are absolute symmetric norms. Then for sufficiently small* $\delta > 0$ *the constant* $k_{1,2}^{(2)}$ *is the infimum of*

$$\mathcal{S}_{1,2} x = \sum_0^{N-1} \frac{|\Delta \tan \theta_t|}{|(\tan \theta_t, 1)|_1 |(1, -\tan \theta_{t+1})|_2^*}$$

subject to the condition $|\Delta \tan \theta_t| \leq \delta |(\tan \theta_t, 1)|_1 |(1, -\tan \theta_{t+1})|_2^*$ *where* $0 = \theta_0 \leq \theta_t \leq \theta_N = \pi/2$.

In Theorem 3.20 it may further be assumed that $\theta_t \leq \theta_{t+1}$ for $t \in \mathcal{D}_{N-1}$ provided $\delta > 0$ is sufficiently small. We provide the details only in the case where $|\cdot|_1$ and $|\cdot|_2$ are Hölder norms.

COROLLARY 3.21. *Let* $|\cdot|_1$ *and* $|\cdot|_2$ *be absolute symmetric norms. Then*

$$\hat{c}_{1,2}^{(2)} = c_{1,2}^{(2)}$$

where $c_{1,2}^{(2)}$ *is the corresponding nonoscillation constant for differential systems.*

PROOF: From Theorem 3.20; if $\delta \to 0^+$ and $\mathcal{P}_N = \{\theta_t : t \in \mathcal{D}_N\}$, then $|\mathcal{P}_{N(\delta)}| \to 0^+$, so that the constant $\hat{c}_{1,2}^{(2)}$ is equal to

$$\liminf_{|\mathcal{P}_N(\delta)|\to 0^+} \sum_0^{N(\delta)-1} \frac{\Delta \tan\theta_t}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*},$$

and therefore

$$\hat{c}_{1,2}^{(2)} = \int_0^{\frac{\pi}{2}} \frac{(\tan\theta)' d\theta}{|(\tan, 1)|_1 |(1, -\tan\theta)|_2^*}.$$

Making the change of variable $s = \tan\theta$ implies (see [1,p.45])

$$\hat{c}_{1,2}^{(2)} = \int_0^{\infty} \frac{ds}{|(s,1)|_1 |(1,-s)|_2^*} = c_{1,2}^{(2)}.$$

□

Define the constant $k_{1,2}^{(2)*}$ to be the infimum of the sum

$$\mathcal{S}_{1,2}^* \boldsymbol{x} = \sum_0^{N-1} \frac{|\Delta \boldsymbol{x}(t)|_1^*}{|\boldsymbol{x}(t)|_2^*}$$

over all locally extremal polygons $\boldsymbol{x} \geq 0$ with respect to the norm $|\cdot|_1^*$ extending from the positive $x_2 -$ axis to the positive $x_1 -$ axis satisfying $|\Delta \boldsymbol{x}(t)|_1^* \leq \delta |\boldsymbol{x}(t)|_2^*$ for $t \in \mathcal{D}_{N-1}$. Let $\boldsymbol{x} \geq 0$ be any locally extremal polygon with respect to the norm $|\cdot|_2$, then

$$\mathcal{S}_{1,2} \boldsymbol{x} = \sum_0^{N-1} \frac{|\Delta \tan\theta_t|}{|(\tan\theta_t, 1)|_1 |(1, -\tan\theta_{t+1})|_2^*}$$

by Proposition 3.19, where θ_t is the angle between the positive $x_2 -$ axis and $\boldsymbol{x}(t)$. Let $\boldsymbol{y}$ be a locally extremal polygon with respect to the norm $|\cdot|_1^*$ such

that $\phi_t = \theta_{N-t}$ is the angle between $y(t)$ and the positive x_2-axis. Then Proposition 3.19 implies

$$\mathcal{S}^*_{1,2}y = \sum_0^{N-1} \frac{|\Delta \tan \phi_t|}{|(\tan \phi_t, 1)|_2^* |(1, -\tan \phi_{t+1})|_1} ,$$

and therefore

$$\mathcal{S}^*_{1,2}y = \sum_0^{N-1} \frac{|\Delta \tan \theta_s|}{|(\tan \theta_{s+1}, 1)|_2^* |(1, -\tan \theta_s)|_1}$$

or

$$\mathcal{S}^*_{1,2}y = \mathcal{S}_{1,2}x .$$

And this with Theorem 3.20, gives the following corollary, which is analogous to the first part of Theorem 5.1 of Friedland [1,p.25-41].

COROLLARY 3.22. *Let* $|\cdot|_1$ *and* $|\cdot|_2$ *be a pair of absolute symmetric norms. Then for sufficiently small* $\delta > 0$,

$$k^{(2)}_{1,2}(N,\delta) = k^{(2)}_{1,2}(N,\delta)^* .$$

It may be shown that, if $|\cdot|_1$ and $|\cdot|_2$ are both absolute symmetric norms, and $|(1,0)|_i = \mu_i$, then

$$\mu_i \max\{|x_1|, |x_2|\} \le |x|_i \le \mu_i(|x_1| + |x_2|)$$

holds for $x \in \mathbb{R}^2$, and therefore

$$(\frac{\mu_2}{\mu_1}) \sum_0^{N-1} \frac{\max\{|\Delta x_1|, |\Delta x_2|\}}{(|x_1| + |x_2|)} \le \mathcal{S}_{1,2}x \le (\frac{\mu_2}{\mu_1}) \sum_0^{N-1} \frac{(|\Delta x_1| + |\Delta x_2|)}{\max\{|x_1|, |x_2|\}} .$$

So, by Theorems 3.4 and 3.14, we have the following corollary.

COROLLARY 3.23. *Let* $|\cdot|_i$ *be an absolute symmetric norm for* $i = 1, 2$; *let* $|(1,0)|_i = \mu_i$; *and let* $0 < \delta \leq \frac{1}{2}$. *Then*

$$\mu_2/\mu_1 \leq k_{1,2}^{(2)}(N,\delta) \leq 2\mu_2/\mu_1.$$

§3.6 The Constant $k_{p_1,p_2}^{(2)}(N,\delta)$

In this section we consider the constant $k_{p_1,p_2}^{(2)}$ where $|\cdot|_{p_1}$ and $|\cdot|_{p_2}$ are Hölder norms.

We define for each $1 \leq p \leq \infty$ the pair of functions

$$C_p\theta = |(1,\tan\theta)|_p^{-1}\ ,\ S_p\theta = \tan\theta|(1,\tan\theta)|_p^{-1}$$

on the domain $[0, \pi/2]$.

THEOREM 3.24. *Let* $|\cdot|_{p_1}$ *and* $|\cdot|_{p_2}$ *be Hölder norms. If* $0 < \delta \leq 1/2$, *then* $k_{p_1,p_2}^{(2)}$ *is the infimum of*

$$S_{p_1,p_2}\boldsymbol{x} = \sum_0^{N-1} C_{p_1}\theta_t C_{q_2}\theta_{t+1}\Delta\tan\theta_t$$

subject to $C_{p_1}\theta_t C_{q_2}\theta_{t+1}\Delta\tan\theta_t \leq \delta$, *where* $0 = \theta_0 \leq \theta_t \leq \theta_{t+1} \leq \theta_N = \pi/2$, *and* $1/p_2 + 1/q_2 = 1$.

PROOF: From Proposition 3.21, the constant $k_{1,2}^{(2)}$ is the infimum of

$$\sum_0^{N-1} C_{p_1}\theta_t C_{q_2}\theta_{t+1}|\Delta\tan\theta_t|$$

subject to those conditions stated. To finish the proof, we need to show that we can make the restriction $\theta_t \le \theta_{t+1}$. This is equivalent to showing that if $\theta_{t+1} \le \theta_t, \theta_{t+2}$, then

$$C_{p_1}\theta_1 C_{q_2}\theta_{t+2}|\tan\theta_t - \tan\theta_{t+2}| \le \sum_{s=t,t+1} C_{p_1}\theta_s C_{q_2}\theta_{s+1}|\Delta\tan\theta_s| .$$

To see this, observe that the right hand side of this inequality as a function of θ_{t+1} on the interval $(0, \min\{\theta_t, \theta_{t+2}\})$ is decreasing, because its derivative

$$-(\tan\theta_{t+1})'\{C_{p_1}\theta_{t+1}C_{q_2}^{1+q_2}\theta_t(1+\tan\theta_t\tan^{q_2-1}\theta_{t+1}) + C_{q_2}\theta_{t+2}C_{p_1}^{1+p_1}\theta_{t+1}(1+\tan\theta_{t+2}\tan^{p_1-1}\theta_{t+1})\}$$

is negative.

□

Reasoning as for Corollary 3.22 we get the following corollary.

COROLLARY 3.25. *Let* $|\cdot|_{p_i}$ *and* $|\cdot|_{q_i}$ *be Hölder norms with* $1/p_i + 1/q_i = 1$ *for* $i = 1, 2$; *and let* $0 < \delta \le \frac{1}{2}$. *Then*

$$k^{(2)}_{p_1,p_2}(N,\delta) = k^{(2)}_{q_2,q_1}(N,\delta) .$$

We now conclude this section with the following theorem.

THEOREM 3.26. *Let* $|\cdot|_{p_1}$ *and* $|\cdot|_{p_2}$ *be a pair of Hölder norms, and let* $0 < \delta \le \frac{1}{2}$. *Then*

$$2^{(\frac{1}{p_2}-\frac{1}{p_1})} - \delta \le k^{(2)}_{p_1,p_2}(N,\delta) \le \delta + 2(1-\delta)^{\frac{1}{2}} .$$

PROOF: From Theorem 3.23, $k^{(2)}_{p_1,p_2}$ is the infimum of

$$S_{p_1,p_2}\boldsymbol{x} = \sum_{0}^{N-1} C_{p_1}\theta_t C_{q_2}\theta_{t+1}\Delta\tan\theta_t$$

subject to

$$C_{p_1}\theta_t C_{q_2}\theta_{t+1}\Delta\tan\theta_t \le \delta \quad \text{for} \quad t \in \mathcal{D}_{N-1} \tag{3.22}$$

where $0 = \theta_0 \le \theta_t \le \theta_{t+1} \le \theta_N = \pi/2$. Let n satisfy $\theta_n \le \pi/4 < \theta_{n+1}$. Then the inequality in (3.22) with $t = n$ along with

$$1 \le 2^{(\frac{1}{p_1}+\frac{1}{q_2})} C_{p_1}\theta_n C_{q_2}\theta_{n+1}\tan\theta_{n+1} \tag{3.23}$$

imply

$$(1 - 2^{(\frac{1}{p_1}+\frac{1}{q_2})}\delta)\tan\theta_{n+1} \le \tan\theta_n$$

and therefore

$$2 - 2^{(\frac{1}{p_1}+\frac{1}{q_2})}\delta \le \tan\theta_n + 1 + \frac{1-\tan\theta_n}{\tan\theta_{n+1}}$$

since $\tan\theta_n \le 1$. Thus

$$2 - 2^{(\frac{1}{p_1}+\frac{1}{q_2})}\delta \le \tan\theta_n + \frac{\Delta\tan\theta_n}{\tan\theta_{n+1}} + \frac{1}{\tan\theta_{n+1}}$$

so that

$$2 - 2^{(\frac{1}{p_1}+\frac{1}{q_2})}\delta \le \sum_{0}^{n-1}\Delta\tan\theta_t + \frac{\Delta\tan\theta_n}{\tan\theta_{n+1}} + \sum_{n+1}^{N-1}\frac{\Delta\tan\theta_t}{\tan\theta_t\tan\theta_{t+1}}\,. \tag{3.24}$$

Because the inequalities

$$1 \le 2^{(\frac{1}{p_1}+\frac{1}{q_2})} C_{p_1}\theta_t C_{q_2}\theta_{t+1},$$

$$1 \le 2^{(\frac{1}{p_1}+\frac{1}{q_2})} C_{p_1}\theta_t C_{q_2}\theta_{t+1} \tan\theta_t \tan\theta_{t+1}$$

hold for $t \in \mathcal{D}_{n-1}$, and $t \in \mathcal{D}_{N-1}\backslash\mathcal{D}_n$ respectively, and the inequality in (3.23) holds, we get

$$\mathcal{S}_{p_1,p_2}\boldsymbol{x} \ge 2^{-(\frac{1}{p_1}+\frac{1}{q_2})}\{\sum_{0}^{n-1}\Delta\tan\theta_t + \frac{\Delta\tan\theta_n}{\tan\theta_{n+1}} + \sum_{n+1}^{N-1}\frac{\Delta\tan\theta_t}{\tan\theta_t\tan\theta_{t+1}}\}\,.$$

Thus $\mathcal{S}_{p_1,p_2}\boldsymbol{x} \ge 2^{(\frac{1}{p_2}-\frac{1}{p_1})} - \delta$ by the inequality in (3.24), and therefore we have $k^{(2)}_{p_1,p_2} \ge 2^{(\frac{1}{p_2}-\frac{1}{p_1})} - \delta$.

To determine the upper bound, observe that the Hölder norm $|\boldsymbol{x}|_p$ as a function of p on $[1,\infty)$ is decreasing, and therefore

$$\frac{|\Delta\boldsymbol{x}|_{p_2}}{|\boldsymbol{x}|_{p_1}} \le \frac{|\Delta\boldsymbol{x}|_1}{|\boldsymbol{x}|_\infty}$$

implies that $k^{(2)}_{p_1,p_2} \le k^{(2)}_{\infty,1}$. So by Theorem 3.14 the proof is complete.

□

CHAPTER IV

THE PROBLEM IN $\mathbb{R}^n$

In Friedland's paper [1, p.48-53], he shows that the constant $c_{1,2}^{(n)}$ does depend on n for many standard norms. In this chapter we conclude that the same is true for the constant $k_{1,2}^{(n)}$.

PROPOSITION 4.1. *Let* $|\cdot|_\infty$ *be the* $p=\infty$ *Hölder norm, and* $|\cdot| = |\cdot|_1$ *be an arbitrary absolute norm. Then*

$$k_{1,\infty}^{(n)}(N,\delta) \le \min_{I_1,I_2} C(I_1,I_2)$$

where I_1 *and* I_2 *are two nonempty complementary subsets of* $\{1,2,\dots,n\}$, *and* $C(I_1,I_2)$ *is the infimum of the sum*

$$\sum_0^{N-1} \frac{\Delta s(t)}{[1+s(t+1)]|w(I_1,I_2,t)|}$$

subject to $\Delta s(t) \le \delta[1+s(t+1)]|w(I_1,I_2,t)|$ *where* $0 = s(0) \le s(t) \le s(t+1) \le s(N) = \infty$, *and the vector function* $w(I_1,I_2,t) = (w_1(t),\dots,w_n(t))$ *is of the form:* $w_i(t) = s(t)$ *for* $i \in I_1$ *and* $w_i(t) = 1$ *for* $i \in I_2$.

PROOF: One reasons analogously to the proof of Theorem 7.4 from Friedland [1, p.50]. Let $x(t) = (x_1(t),\dots,x_n(t)) \in V^{(n)}$ be such that $x_i(0) = 0$ if $i \in I_1$, and $x_i(N) = 0$ if $i \in I_2$ for some I_1 and I_2 as above. Let

$$u(t) = \max_{i\in I_1} |x_i(t)|,$$

$$v(t) = \max_{i\in I_2} |x_i(t)|.$$

Then the vector $y(t)$ defined by $y_i(t) = u(t)$ for $i \in I_1$, and $y_i(t) = v(t)$ for $i \in I_2$ satisfies

$$|\Delta y(t)|_\infty \le |\Delta x(t)|_\infty,$$

$$|y(t)| \ge |x(t)|$$

as $|\cdot|$ is an absolute norm. Thus

$$\sum_0^{N-1} \frac{|\Delta x|_\infty}{|x|} \ge \sum_0^{N-1} \frac{|\Delta y|_\infty}{|y|}.$$

Putting $s(t) = u(t)/v(t)$ and noting that

$$|\Delta s(t)| = \left|\frac{u(t+1)\Delta v(t) - v(t+1)\Delta u(t)}{v(t+1)v(t)}\right|$$

along with

$$\left|\sum_{i=1}^{2} a_i b_i\right| \le (|a_1| + |a_2|) \max(|b_1|, |b_2|)$$

implies

$$|\Delta s(t)| \le \frac{[u(t+1) + v(t+1)]|\Delta y(t)|_\infty}{v(t+1)v(t)},$$

we get that

$$\sum_0^{N-1} \frac{|\Delta y|_\infty}{|y|} \ge \sum_0^{N-1} \frac{|\Delta s(t)|}{[1 + s(t+1)]|w(I_1, I_2, t)|}. \tag{4.1}$$

To finish the proof, we show that the equality sign in (4.1) can hold for any choice of $s(t)$, $t \in \mathcal{D}_N$. If y is such that $u(t) = \eta(t)$ and $v(t) = 1 - \eta(t)$ where $0 = \eta(0) \le \eta(t) \le \eta(t+1) \le \eta(N) = 1$, then making the substitution $\ell(t) = [1 - \eta(N-t)]/\eta(N-t)$ in

$$\sum_0^{N-1} \frac{|\Delta y(t)|_\infty}{|y(t)|} = \sum_0^{N-1} \frac{\Delta \eta(t)}{|y(t)|}$$

implies

$$\sum_0^{N-1} \frac{|\Delta y(t)|_\infty}{|y(t)|} = \sum_0^{N-1} \frac{\Delta \ell(t)}{[1+s(t+1)]|w(I_1, I_2, t)|}.$$

Choosing $\ell(t) = s(t)$ gives the required equality in (4.1).

□

THEOREM 4.2. *Let* $|\cdot|_p$ *be the Hölder norm for* $1 \le p \le \infty$ *and* $0 < \delta \le 1$. *Then*

$$k_{p,\infty}^{(n)} \le [(n-1)/2]^{-\frac{1}{p}} k_{p,\infty}^{(2)} \quad \text{for} \quad n \ge 2,$$

where [ ] *is the greatest integer function.*

PROOF: Set $I_1 = \{1, \ldots, m\}$ and $I_2 = \{m+1, \ldots, n\}$ in Proposition 4.1. Then $k_{p,\infty}^{(n)}$ is bounded above by the sum

$$\sum_0^{N-1} \frac{\Delta s(t)}{[1+s(t+1)][ms^p(t)+n-m]^{\frac{1}{p}}} \tag{4.2}$$

where $\Delta s(t) \le \delta[1+s(t+1)][ms^p(t)+n-m]^{\frac{1}{p}}$ and $0 = s(0) \le s(t) \le s(t+1) \le s(N) = \infty$. If $n = 2m$, then setting $\tan\theta_t = s(t)$ in (4.2) implies $k_{p,\infty}^{(2m)} \le m^{-\frac{1}{p}} k_{p,\infty}^{(2)}$ by Theorem 3.20. If $n = 2m+1$, then $k_{p,\infty}^{(2m+1)} \le k_{p,\infty}^{(2m)}$ implies $k_{p,\infty}^{(2m+1)} \le m^{-\frac{1}{p}} k_{p,\infty}^{(2)}$.

□

PROPOSITION 4.3. *Let* $|\cdot|_{p_1}$ *and* $|\cdot|_{p_2}$ *be Hölder norms; let* $p_2 \le p_1$; *and let* $0 < \delta \le 1$. *Then* $k_{p_1,p_2}^{(n)}$ *is the infimum of*

$$S_{p_1,p_2} x = \sum_0^{N-1} \frac{|\Delta x|_{p_2}}{|x|_{p_1}} \tag{4.3}$$

over all $x \in V^{(n)}$ *such that* $x_i(0)x_i(N) = 0$ *for* $i = 1, 2, \ldots, n$.

PROOF: Let $x \in V^{(n)}$. If $x_i(N-1)x_i(N) \leq 0$ then changing $x_i(N)$ to zero decreases the sum in (4.3). So assume $x_i(N) = 0$ if $x_i(N-1)x_i(N) \leq 0$. Now, since $\delta \leq 1$, there exists $j \in \{1, \ldots, n\}$ such that $x_i(0) \neq 0$ and $x_i(N) = 0$. Assume $j = n$. Let $i \neq n$ be such that $x_i(N) \neq 0$, and let

$$\ell = \inf\{t \geq 1 : \; x_i(t)x_i(t-1) \leq 0\}.$$

Then for $t = \mathcal{D}_{\ell-1}$, x may be written as

$$x(t) = \left(a_1(t), \ldots, a_i(t), \varepsilon r_t C_{p_1}\theta_t, a_{i+1}(t), \ldots, a_{n-1}(t), \bar{\varepsilon} r_t S_{p_1}\theta_t\right)$$

and for $t = \ell$ as

$$x(\ell) = \left(a_1(\ell), \ldots, a_i(\ell), \varepsilon r_\ell C_{p_1}\theta_\ell, a_{i+1}(\ell), \ldots, a_{n-1}(\ell), \bar{\varepsilon} r_\ell S_{p_1}\theta_\ell\right)$$

where $\varepsilon, \bar{\varepsilon} = \pm 1$; $r_t > 0$; and $\theta_t \in [0, \pi/2]$. To finish the proof we show that the quotient

$$\frac{|\Delta x(t)|_{p_2}}{|x(t)|_{p_1}}$$

decreases if we replace θ_t by $\pi/2$ for $t \in \mathcal{D}_{\ell-1}$. This is equivalent to showing

$$|\Delta[r_t(S_{p_1}\theta_t, C_{p_1}\theta_t)]|_{p_2} \geq |\Delta r_t|_{p_2} \quad \text{for} \quad t \in \mathcal{D}_{\ell-2}, \quad \text{and} \tag{4.4}$$

$$|\zeta_1 - \zeta_2|_{p_2} \geq |\zeta_3 - \zeta_2|_{p_2} \tag{4.5}$$

where $\zeta_1 = r_{\ell-1}(-C_{p_1}\theta_{\ell-1}, S_{p_1}\theta_{\ell-1})$, $\zeta_2 = r_\ell(C_{p_1}\theta_\ell, S_{p_1}\theta_\ell)$, and $\zeta_3 = r_{\ell-1}(0, 1)$.

To show the inequalities in (4.4), it suffices to show

$$|\xi_1 - \xi_2|_{p_2} \geq |\xi_3 - \xi_4|_{p_2} \tag{4.6}$$

where $\xi_1 = r(C_{p_1}\theta, S_{p_1}\theta)$, $\xi_2 = (C_{p_1}\phi, S_{p_1}\phi)$, $\xi_3 = (0, r)$, and $\xi_4 = (0, 1)$ as illustrated in Figure 4.7.

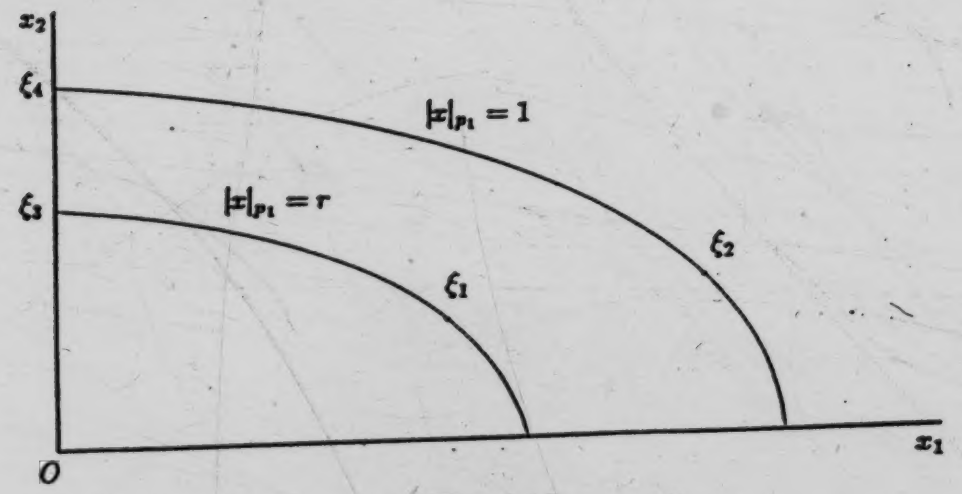

Figure 4.7

To show (4.6), observe that $|x|_{p_2} \geq |x|_{p_1}$ together with

$$|\xi_1 - \xi_2|_{p_1} \geq |\xi_3 - \xi_4|_{p_1} = |\xi_3 - \xi_4|_{p_2}$$

imply

$$|\xi_1 - \xi_2|_{p_2} \geq |\xi_3 - \xi_4|_{p_2}$$

which is (4.6).

To show (4.5), let ζ_4 be the point of intersection between the edge $[\zeta_1, \zeta_2]$ and the x_2-axis as illustrated in Figure 4.8.

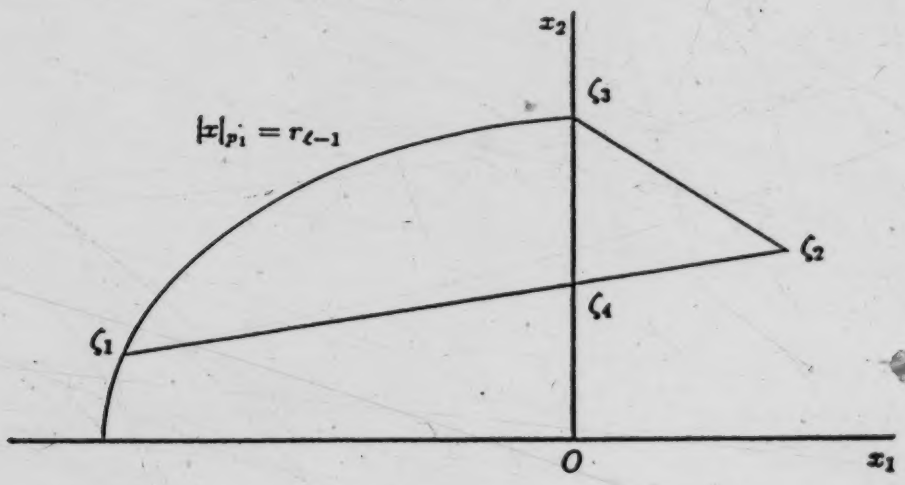

Figure 4.8

Then $|x|_{p_2} \geq |x|_{p_1}$ and

$$|\zeta_1 - \zeta_4|_{p_1} \geq |\zeta_3 - \zeta_4|_{p_1} = |\zeta_3 - \zeta_4|_{p_2}$$

implies

$$|\zeta_1 - \zeta_4|_{p_2} \geq |\zeta_3 - \zeta_4|_{p_2} ,$$

and therefore

$$|\zeta_1 - \zeta_2|_{p_2} \geq |\zeta_3 - \zeta_4|_{p_2} + |\zeta_4 - \zeta_2|_{p_2}$$

since $|\zeta_1 - \zeta_2|_{p_2} = |\zeta_1 - \zeta_4|_{p_2} + |\zeta_4 - \zeta_2|_{p_2}$. But this implies (4.5) because

$$|\zeta_3 - \zeta_4|_{p_2} + |\zeta_4 - \zeta_2|_{p_2} \geq |\zeta_3 - \zeta_2|_{p_2}.$$

□

THEOREM 4.4. *Let* $|\cdot|_{p_1}$ *and* $|\cdot|_{p_2}$ *be Hölder norms where* $p_2 \le p_1$. *Then*

$$k^{(2)}_{p_1,p_2}(N,\delta) = k^{(n)}_{p_1,p_2}(N,\delta)$$

for $n \ge 2$, *and* $0 < \delta \le 1$.

PROOF: From Proposition 4.3, it suffices to show that if x is an oscillatory function in $\mathbb{R}^n$ such that $x_i(0)x_i(N) = 0$ for $1 \le i \le n$, then there exists an oscillatory vector function y in $\mathbb{R}^2$ such that

$$\frac{|\Delta y(t)|_{p_2}}{|y(t)|_{p_1}} \le \frac{|\Delta x(t)|_{p_2}}{|x(t)|_{p_1}}$$

for $t \in \mathcal{D}_{N-1}$. Let the set $\{1,\dots,n\}$ be decomposed into two complementary sets I_1 and I_2. That is $x_i(0) = 0$ if $i \in I_1$ and $x_i(N) = 0$ for $i \in I_2$. Define the vector functions $\varepsilon(t)$ and $\eta(t)$ by

$$\varepsilon_i(t) = x_i(t), \quad \eta_i(t) = 0 \qquad \text{for } i \in I_1,$$

$$\varepsilon_j(t) = 0, \qquad \varepsilon_j(t) = x_j(t) \quad \text{for } j \in I_2.$$

Let

$$y_1(t) = |\varepsilon(t)|_{p_1}, \qquad y_2(t) = |\eta(t)|_{p_1}$$

$$z_1(t) = |\Delta\varepsilon(t)|_{p_2}, \qquad z_2(t) = |\Delta\eta(t)|_{p_2}.$$

Since $|x|_{p_2} \ge |x|_{p_1}$ for $p_2 \le p_1$ we obtain

$$z_1(t) \ge |\Delta\varepsilon(t)|_{p_1} \quad , \quad z_2(t) \ge |\Delta\eta(t)|_{p_1}.$$

Thus the triangle inequalities

$$|\Delta\varepsilon(t)|_{p_1} \ge \big|\Delta|\varepsilon(t)|_{p_1}\big| \quad , \quad |\Delta\eta(t)|_{p_1} \ge \big|\Delta|\eta(t)|_{p_1}\big|$$

imply that $z_i(t) \ge |\Delta y_i(t)|$ for $i = 1, 2$, and therefore we conclude

$$\frac{|\Delta x(t)|_{p_2}}{|x(t)|_{p_1}} = \frac{|(z_1(t), z_2(t))|_{p_2}}{|(y_1(t), y_2(t))|_{p_1}} \ge \frac{|\Delta(y_1(t), y_2(t))|_{p_2}}{|(y_1(t), y_2(t))|_{p_1}}$$

where y is oscillatory.

□

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